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(continued after index)

Gabor Toth

Finite Möbius Groups, Minimal Immersions of Spheres, and Moduli



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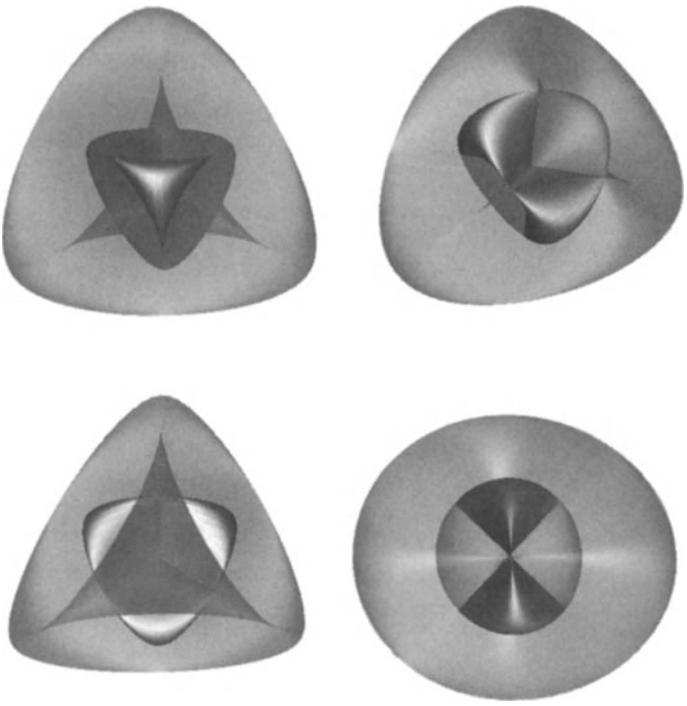
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On September 1, 1944, the railway station of a small Hungarian town collapsed during a bombing raid. An 18-year old girl barely alive and with a fractured skull was taken out of the rubble. Not only did she survive but went to medical school and became a leading surgeon of her hometown. She continued her practice until the age of 75. With utmost respect and love, this book is dedicated to her, Dr. Gabriella Török, whom I have come to know as my mother.



Moduli for $SU(2)$-equivariant quadratic eigenmaps from S^3.

Introduction and Synopsis

"Spherical soap bubbles," isometric minimal immersions of round spheres into round spheres, or spherical minimal immersions for short, belong to a fast growing and fascinating area between algebra and geometry. This theory has rich interconnections with a variety of mathematical disciplines such as invariant theory, convex geometry, harmonic maps, and orthogonal multiplications.

In the spring of 1994, I began writing up some notes centered around the DoCarmo-Wallach classification theory of spherical minimal immersions, and hoped that one day a coherent picture would emerge. Since then I kept adding more material to my notes, and it soon became apparent that the accumulated text could fill the pages of a book.

Spherical minimal immersions can be discussed restrictively in their own realm, but they can also be treated imbedded in beautiful classical mathematics. I have chosen the latter (and longer) path. In trying to make this monograph accessible not just to research mathematicians but mathematics graduate students as well, I included sizeable pieces of material from upper level undergraduate mathematics courses. Since 3-dimensional Euclidean geometry is often neglected in the undergraduate mathematics curriculum, I have chosen to start with some basic material on the five Platonic solids, and go on to the classification of finite groups of rotations in $\mathbf{R}^3$, and the classification of finite Möbius groups. The first few sections of Chapter 1 take off where my *Glimpses of Algebra and Geometry* (Springer-Verlag, New York, 1998) ended, with occasional overlaps. In trying to keep the prerequisites to a minimum, especially in the first half of the book, I used only standard material from undergraduate mathematics courses, notably

from calculus, geometry, and abstract algebra. More advanced material and related references appear occasionally. In trying to separate these from the main text I placed them in "Remarks." Taking the longer path also resulted in the "Additional Topics" sections at the end of each chapter. The most prominent and detailed covers Felix Klein's classic treatise on the icosahedron. To limit the size of the book I compressed some easier material into problems at the end of each chapter. Some of these are still challenging, and this explains the occasionally more than generous hints attached to the more involved problems.

I believe that a proper introduction to the subject is to give a short guided tour, and, in doing so here, I take the liberty to leave the full explanation (and all references) to the main text.

The theory of spherical soap bubbles studies isometric minimal immersions of round spheres into round spheres of different dimensions. The principal object to study is an isometric minimal immersion $f : S^m_\kappa \to S_V$ of the Euclidean m-sphere of constant curvature κ into the unit sphere S_V (of curvature 1) of a Euclidean vector space V. (It is of technical convenience to scale the spherical range S_V to curvature 1. In addition, we keep the linear range V arbitrary since in naturally occurring examples there is no a priori basis in V.)

A classical example is the Veronese minimal immersion $Ver : S^2_{1/3} \to S^4$. It factors through the antipodal map in $S^2_{1/3}$ and gives an isometric minimal imbedding of the real projective plane $\mathbf{R}P^2$ (of curvature $1/3$) into the 4-sphere S^4 as the Veronese minimal surface (Figure 1 shows various linear projections of the surface viewed from S^4 first stereographically projected to $\mathbf{R}^4$).

The general theory first took off in 1966 when Takahashi proved that an isometric minimal immersion $f : S^m_\kappa \to S_V$ exists iff $\kappa = m/\lambda_p$, where $\lambda_p = p(p+m-1)$ is the p-th eigenvalue of the spherical Laplacian $\triangle^{S^m}$ on S^m. He also showed that, in this case, the components of f are spherical harmonics of order p on S^m (with the curvature 1 metric), i.e., they are eigenfunctions of $\triangle^{S^m}$ with eigenvalue λ_p. To maintain uniformity, it is convenient to keep the curvature 1 metric on the domain S^m. Our immersions then become conformal (with conformality factor λ_p/m), and we arrive at the concept of a spherical minimal immersion $f : S^m \to S_V$ of degree p.

A rich variety of spherical minimal immersions can be obtained by the "equivariant construction," first used in this context by Mashimo in 1984, and then fully exploited by DeTurck and Ziller in 1992. These spherical minimal immersions are orbit maps $f_\xi : S^3 \to S_W$, where W is a representation space of the special unitary group $SU(2)$. (W is endowed with an $SU(2)$-invariant scalar product, and ξ is of unit length in W.) The image of f_ξ is the $SU(2)$-orbit through ξ. f_ξ is a spherical minimal immersion iff this $SU(2)$-orbit is minimal. Special choices of W give minimum codimen-

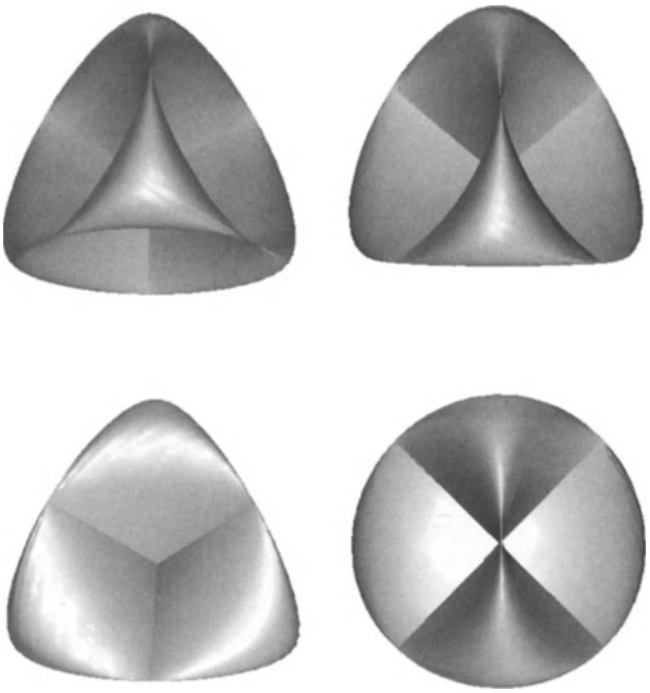


Figure 1.

sion examples such as Mashimo's degree 6 spherical minimal immersion $f : S^3 \to S^6$.

In fact, as J. D. Moore showed in 1976, any spherical minimal immersion $f : S^m \to S_V$ of degree ≥ 2 satisfies $\dim V \geq 2m + 1$.

Mashimo's example shows that Moore's general lower bound is the best possible. This settled a number of suggestions made earlier by Chern, DoCarmo-Wallach, and others.

Choosing ξ as one of Klein's polynomial invariants for a finite subgroup $G \subset SU(2)$ allows the orbit map f_ξ to be factored into a spherical minimal imbedding of the orbit space S^3/G into a sphere. G is either cyclic, or dihedral, or a binary polyhedral group (the twofold cover of the group of symmetries of a Platonic solid along the twofold covering projection $\pi : SU(2) \to SO(3)$). The orbit space S^3/G is either a lens space, a dihedral manifold, or a polyhedral manifold.

A careful enumeration of the possible cases led DeTurck and Ziller to conclude that all homogeneous spherical space forms can be isometrically and minimally imbedded into spheres. This is the main result in Chapter 1.

Since Klein's theory of the icosahedron is developed in this chapter as a technical tool, we include here a solution of the irreducible quintic in terms of radicals and hypergeometric series. The Additional Topic at the end of

Chapter 1 is a somewhat simplified treatment of the main results of Klein's Icosahedron Book (not using the fairly technical Brioschi quintic).

By Takahasi's result, the problem of "classifying" spherical minimal immersions $f : S^m \to S_V$ has two natural parameters; the domain dimension m, and the degree p (the range V is kept arbitrary).

To construct a reasonable "moduli space", we make two reductions. First, we assume that $f : S^m \to S_V$ is full. Geometrically, the image of f is not contained in a proper great sphere of S_V. Second, we do not distinguish between congruent maps; maps that differ by an isometry between their ranges.

The universal example for a spherical minimal immersion of degree p is the standard minimal immersion $f_p : S^m \to S_{\mathcal{H}^p}$, where $\mathcal{H}^p$ is the space of all spherical harmonics $\mathcal{H}^p$ of order p on S^m. It is uniquely defined (up to congruence) by the requirement that, relative to a (suitably scaled) L_2-orthonormal basis in $\mathcal{H}^p$, the components of f_p are orthonormal. The Veronese map $Ver : S^2 \to S^4$ mentioned above is actually the standard minimal immersion in degree 2. In fact, as follows from the work of Calabi (1967) and DoCarmo-Wallach (1970), up to congruence, the standard minimal immersion $f_p : S^2 \to S_{\mathcal{H}^p} = S^{2p}$ of degree p is the only spherical minimal immersion for $m = 2$.

This rigidity result is the starting point of our study of spherical minimal immersions. The natural question, posed by DoCarmo and Wallach in the early 1970s, is to what extent is the standard minimal immersion unique among all spherical minimal immersions of the same degree. In addition, if nonuniqueness occurs, what is the structure of the corresponding "moduli space." In 1971, DoCarmo and Wallach proved that the standard minimal immersion is unique (up to congruence) for $p \leq 3$. The main aim of their work, however, was to show that, for $m \geq 3$ and $p \geq 4$, unicity fails, and, indeed, the set of (congruence classes of) full spherical minimal immersions $f : S^m \to S_V$ of degree p can be parametrized by a moduli space $\mathcal{M}^p$. The moduli space is a compact convex body in a finite dimensional vector space $\mathcal{F}^p$ of dimension ≥ 18. We let $\langle f \rangle \in \mathcal{M}^p$ denote the parameter point corresponding to the spherical minimal immersion $f : S^m \to S_V$ under the DoCarmo-Wallach parametrization. $SO(m+1)$ acts on $\mathcal{M}^p$ via precomposition $g \cdot \langle f \rangle = \langle f \circ g^{-1} \rangle$, $\langle f \rangle \in \mathcal{M}^p$, $g \in SO(m+1)$. It turns out that this action extends to the linear span $\mathcal{F}^p$ of $\mathcal{M}^p$, and $\mathcal{F}^p$ becomes a linear $SO(m+1)$-representation space.

The problem of determining the dimension $\dim \mathcal{M}^p = \dim \mathcal{F}^p$ now becomes more tractable since all we need to do is to decompose $\mathcal{F}^p$ into irreducible components, a standard problem in representation theory (other than the problem of translating minimality into representation theoretical data).

The space of spherical harmonics $\mathcal{H}^p$ is also a representation space for $SO(m+1)$, where the action of $SO(m+1)$ is given by precomposition: $g \cdot \chi = \chi \circ g^{-1}$, $\chi \in \mathcal{H}^p$, $g \in SO(m+1)$. This representation extends to the

full tensor algebra over $\mathcal{H}^p$. In particular, the symmetric square $S^2(\mathcal{H}^p)$, the space of symmetric endomorphisms of $\mathcal{H}^p$, is an $SO(m+1)$-representation. It turns out that $\mathcal{F}^p$ is a subrepresentation of $S^2(\mathcal{H}^p)$.

Cartan's theory tells us that a complex irreducible representation of $SO(n)$ is determined by its highest weight vector v. With respect to the standard maximal torus in $SO(n)$ (providing a coordinate system for the Cartan subalgebra), v becomes an $[n/2]$-tuple $v = (v_1, \ldots, v_{[n/2]}) \in \mathbf{Z}^{[n/2]}$, $[n/2] = \operatorname{rank}(SO(n))$. We denote this representation by V^v. As an example, we have

$$\mathcal{H}^q = V^{(q,0,\ldots,0)}$$

as complex representations of $SO(n)$. Here $\mathcal{H}^q$ is the space of complex valued spherical harmonics of order q on S^m.

With this, our problem becomes twofold:

(1) What are the highest weights of the irreducible $SO(m+1)$-subrepresentations that occur (with possible multiplicity) in $S^2(\mathcal{H}^p)$, or, more generally, in $\mathcal{H}^p \otimes \mathcal{H}^q$?

(2) Which of the highest weights in $S^2(\mathcal{H}^p)$ occur in the $SO(m+1)$-subrepresentation $\mathcal{F}^p$?

In 1971 Wallach gave an affirmative answer for (1). In fact, with the notations above, we have

$$\mathcal{H}^p \otimes \mathcal{H}^q = \sum_{(u,v)\in\triangle_0^{p,q};\, u+v\equiv p+q \pmod 2} V^{(u,v,0,\ldots,0)}, \quad p \geq q \geq 1, m \geq 3, \tag{1}$$

as complex representations, where $\triangle_0^{p,q}$ is the closed convex triangle in $\mathbf{R}^2$ with vertices $(p-q,0)$, (p,q) and $(p+q,0)$.

For $p = q$, $\mathcal{H}^p \otimes \mathcal{H}^p = S^2(\mathcal{H}^p) \oplus \wedge^2(\mathcal{H}^p)$, and it turns out that the irreducible subrepresentations in the skew-symmetric part $\wedge^2(\mathcal{H}^p)$ have highest weights with odd integral components. Deleting these from (1) we obtain

$$S^2(\mathcal{H}^p) = \sum_{(u,v)\in\triangle_0^p;\, u,v \text{ even}} V^{(u,v,0,\ldots,0)}, \tag{2}$$

where we set $\triangle_0^p = \triangle_0^{p,p}$.

The main result of DoCarmo-Wallach is the lower estimate

$$\mathcal{F}^p \otimes_{\mathbf{R}} \mathbf{C} \supset \sum_{(u,v)\in\triangle_2^p;\, u,v \text{ even}} V^{(u,v,0,\ldots,0)}, \quad p \geq 4,\ m \geq 3, \tag{3}$$

where $\triangle_2^p$ is the closed convex triangle in $\mathbf{R}^2$ with vertices $(4,4)$, (p,p) and $(2(p-2),4)$. The Weyl dimension formula, an explicit formula for the dimension of an irreducible representation in terms of its highest weight, now gives the lower bound

$$\dim \mathcal{M}^p \geq \dim \mathcal{M}^4 \geq 18, \quad p \geq 4,\ m \geq 3,$$

where we used (3). DoCarmo and Wallach believed that their lower bound is sharp so that actually equality holds in (3). The positive resolution of this "exact dimension conjecture" is the main result of Chapter 3.

Ignoring the condition of conformality in the definition of spherical minimal immersions, we arrive at the concept of an eigenmap. A p-eigenmap $f : S^m \to S_V$ is simply a map whose components are spherical harmonics on S^m of order p. Much of the DoCarmo-Wallach theory for spherical minimal immersions can be adapted to p-eigenmaps. It turns out that the moduli space $\mathcal{L}^p$ of p-eigenmaps $f : S^m \to S_V$ spans an $SO(m+1)$-subrepresentation $\mathcal{E}^p \subset S^2(\mathcal{H}^p)$, and we have the decomposition

$$\mathcal{E}^p \otimes_{\mathbf{R}} \mathbf{C} = \sum_{(u,v)\in\triangle_1^p;\, u,v \text{ even}} V^{(u,v,0,\ldots,0)}, \quad p \geq 2,\ m \geq 3, \tag{4}$$

where $\triangle_1^p$ is the closed convex triangle in $\mathbf{R}^2$ with vertices $(2,2)$, (p,p) and $(2(p-1),2)$.

We also have

$$\mathcal{M}^p = \mathcal{L}^p \cap \mathcal{F}^p,$$

i.e. the moduli space $\mathcal{M}^p$ for spherical minimal immersions is a linear slice of the moduli space $\mathcal{L}^p$ for eigenmaps. The construction of $\mathcal{L}^p$ is technically simpler than the construction of $\mathcal{M}^p$. For this reason, we treat eigenmaps in Chapter 2 to prepare for the more involved material in Chapter 3.

The lowest nonrigid domain dimension $m = 3$ is special in the sense that $SO(4) = SU(2) \cdot SU(2)'$, with each $SU(2)$ acting transitively on S^3. Hence, just as in the equivariant construction, we can consider the $SU(2)$- and $SU(2)'$-equivariant spherical minimal immersions, whose moduli spaces are the fixed point sets $(\mathcal{M}^p)^{SU(2)}$ and $(\mathcal{M}^p)^{SU(2)'}$.

The components of the linear span $(\mathcal{F}^p)^{SU(2)}$ of the moduli space $(\mathcal{M}^p)^{SU(2)}$ parametrizing $SU(2)$-equivariant spherical minimal immersions $f : S^3 \to S_V$ are distributed along the northwestern edge of the triangle $\triangle_2^p$ in (3). These moduli are perhaps the least subtle to analyze. The case of quartic spherical minimal immersions $f : S^3 \to S_V$ with 18-dimensional moduli space is fully detailed in Section 3.6.

The finer technical details of the proof of the exact dimension conjecture require a novel approach that uses operators defined on eigenmaps and minimal immersions. The simplest of these are the degree raising and lowering operators. They associate to a p-eigenmap $f : S^m \to S_V$ $(p \pm 1)$-eigenmaps $f^{\pm} : S^m \to S_{V \otimes \mathcal{H}^1}$. Between the moduli spaces, the correspondences $f \mapsto f^{\pm}$ give rise to homomorphisms $\Phi^{\pm} : S^2(\mathcal{H}^p) \to S^2(\mathcal{H}^{p\pm1})$ of representations of $SO(m+1)$. They satisfy $\Phi^{\pm}(\mathcal{L}^p) \subset \mathcal{L}^{p\pm1}$ and $\Phi^{\pm}(\mathcal{M}^p) \subset \mathcal{M}^{p\pm1}$. In terms of the decomposition (2), Φ^+ is injective and corresponds to the inclusion $\triangle_0^p \subset \triangle_0^{p+1}$. Φ^- is surjective with kernel distributed along the northeastern edge of $\triangle_0^p$. The former gives equivariant imbeddings of the moduli $\mathcal{L}^p$ into $\mathcal{L}^{p+1}$, and $\mathcal{M}^p$ into $\mathcal{M}^{p+1}$. The latter gives lower bounds on the range of eigenmaps that correspond to points in $\ker \Phi^-$. For ex-

ample, for quartic minimal immersions $f : S^m \to S_V$, $m \geq 4$, we obtain $\dim V \geq m(m+5)/6$, a quadratic lower bound that improves Moore's linear lower bound.

A general operator associating a q-eigemap to a p-eigenmap is determined by an irreducible component W of $\mathcal{H}^p \otimes \mathcal{H}^q$. The operator associated to W reflects the symmetries of the Young tableau of W. A prominent example is the operator of infinitesimal rotations that associates to a p-eigenmap $f : S^m \to S_V$ a p-eigenmap $\hat{f} : S^m \to S_{V \otimes so(m+1)^*}$ whose components are obtained by infinitesimally rotating the components of f in the coordinate planes of $\mathbf{R}^{m+1}$. (Here $W = so(m+1)^*$ with the coadjoint representation.) The corresponding operator $\mathcal{A}_p$ induces a self-map of the moduli $\mathcal{M}^p$. The eigenvalues of $\mathcal{A}_p$ on the irreducible components of $\mathcal{E}^p$ can be computed since, up to a suitable affine transformation, $\mathcal{A}_p$ is essentially given by the Casimir operator. It turns out that $\mathcal{A}_p$ is a contraction on $\mathcal{M}^p$, and suitable (and computable) iterates of $\mathcal{A}_p$ bring all boundary points of $\mathcal{M}^p$ into the interior of $\mathcal{M}^p$. Since the interior corresponds to maximal range dimension, we obtain a variety of lower bounds for the range of the original spherical minimal immersion. This is the subject of Chapter 4.

Many of the results in this book can be extended (in various degrees of generality) to eigenmaps and spherical minimal immersions $f : G/K \to S_V$, where G/K is a compact Riemannian homogeneous space. The case when G/K is a compact rank-one symmetric space is particularly suitable for further study since the spectrum of the Laplacian $\triangle^{G/K}$ is explicitly known. There are two significant obstacles to carrying out the entire study, however. First, for a non-spherical compact rank-one symmetric space G/K, the symmetric square of an eigenspace of the Laplacian no longer has multiplicity 1 decomposition into irreducible G-modules. This implies that the DoCarmo-Wallach lower bound for the dimension of the moduli space is no longer sharp. (For details, see Toth [7].) The second (technical) obstacle is the lack of conformal fields on G/K. In the spherical case $S^m = SO(m+1)/SO(m)$, conformal fields play a crucial role in the proof of the exact dimension conjecture.

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1
Finite Möbius Groups

1.1 Platonic Solids and Finite Rotation Groups

The purpose of this introductory section is to classify all finite isometry groups G acting on $\mathbf{R}^3$. Restricting ourselves first to direct (orientation preserving) isometries, using a Burnside counting argument, we will prove a result of Klein [1] asserting that a finite group G of direct isometries of $\mathbf{R}^3$ is either cyclic, dihedral, or the symmetry group of a Platonic solid. We finish this section augmenting G by opposite (orientation reversing) isometries. The main reference for this section is Coxeter [2].

We begin with the following result of Euler:

Theorem 1.1.1. *A (nontrivial) direct isometry S of $\mathbf{R}^3$ that fixes a point is a rotation whose axis passes through the fixed point.*

PROOF. We give here a somewhat unusual proof based on the spherical triangle inequality. (A proof using linear algebra is outlined in Problem 1.1. For a generalization, see Problem 1.2.)

We may assume that the origin is a fixed point of S. (Otherwise, we consider the conjugate $T_v^{-1} \circ S \circ T_v$ of S, where T_v is the translation with vector v connecting the origin and a fixed point of S.) Since S fixes the origin, it must be linear. (On the one hand, an isometry of $\mathbf{R}^3$ is completely determined by its action on four noncoplanar points. On the other hand, given two congruent tetrahedra with a common vertex at the origin, there is a unique linear transformation that carries one of the tetrahedra to the other with the vertices corresponding to each other according to the pre-

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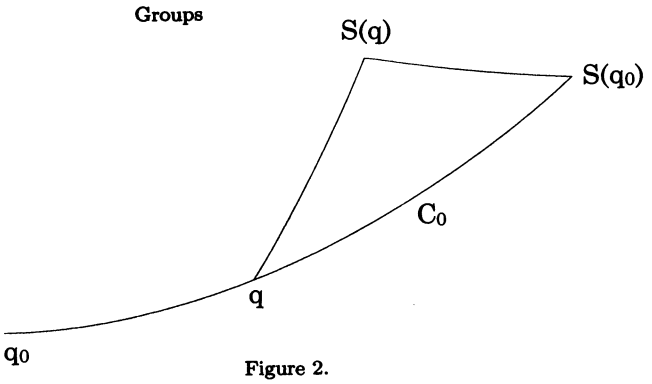


Figure 2.

scribed congruence.) Since S is a linear isometry, it leaves the unit sphere S^2 invariant. We now claim that S leaves a great circle on S^2 invariant. To show this consider the *translation distance function* $f : S^2 \to [0, \pi]$ of S, defined by

$$f(p) = d(p, S(p)), \quad p \in S^2,$$

where d is the spherical distance function on S^2. Note that $f(p_0) = 0$ iff $S(p_0) = p_0$, and $f(p_0) = \pi$ iff $S(p_0) = -p_0$. In either case, since S is linear, (the linear span of) the great circle $C \subset S^2$ perpendicular to the vectors $\pm p_0$ (in $\mathbf{R}^3$) is invariant under S.

Let $q_0 \in S^2$ be a point at which f attains a minimum. (In the following argument we will only use that f has local minimum at q_0.) By the previous note, we may assume that $0 < f(q_0) < \pi$. Let $C \subset S^2$ be the great circle passing through q_0 and $S(q_0)$. We now show that S leaves C invariant.

Let $C_0 \subset C$ be the (open) shorter great circular arc connecting q_0 and $S(q_0)$, and let $q \in C_0$ close to q_0. We have $S(q) \in C$. Indeed, if $S(q) \notin C$ then, by the triangle inequality applied to the spherical triangle with vertices q, $S(q)$, $S(q_0)$ (and side lengths $< \pi$), we have

$$\begin{aligned} f(q_0) = d(q_0, S(q_0)) &= d(q_0, q) + d(q, S(q_0)) \\ &= d(S(q_0), S(q)) + d(q, S(q_0)) \\ &> d(q, S(q)) = f(q). \end{aligned}$$

(See Figure 2.) This is in contradiction to the fact that f attains a minimum at q_0. Thus, for any $q \in C_0$ close to q_0, we have $S(q) \in C$. Since spherical distances add up along C, we see that S leaves C invariant. The claim follows.

Let $V \subset \mathbf{R}^3$ be the linear span of C, and $\pm p_0 \in S^2$ perpendicular to V. Since S is linear, it leaves the 2-dimensional plane V invariant, and $S(p_0) = \pm p_0$. By the classification of linear isometries in $\mathbf{R}^2$, we know that $S|_V$ is either a rotation or a reflection. Since S is direct in $\mathbf{R}^3$, in the first case we have $S(p_0) = p_0$, so that S is a rotation with axis $\mathbf{R} \cdot p_0$. In the

second case $S(p_0) = -p_0$ (again since S is direct), and we see that S is a half-turn, and the axis coincides with the axis of the reflection $S|_V$. The theorem follows.

Remark. For a generalization of the argument in the proof, see Ozols [1]. Note also that the translation distance function plays a crucial role in the classification of isometries in hyperbolic space; see Thurston [1], pp. 94-96, and also Helgason [1], pp. 278-279.

Let us now recall a few facts about Platonic solids and their symmetry groups:

Let $P \subset \mathbf{R}^3$ be a convex polyhedron. Let V denote the number of vertices, E the number of edges, and F the number of faces of P. By Euler's famous theorem on convex polyhedra, we have

$$V - E + F = 2. \tag{1.1.1}$$

(This formula was supposedly known to Archimedes, and certainly to Descartes. There are many ingenious proofs. The shortest proof generalizes (1.1.1) to connected planar graphs, and builds up the graph step-by-step; see Coxeter [2]. For a beautiful geometric proof due to von Staudt, see Coxeter [1]. For yet another proof using spherical geometry, see Problem 1.3.)

An isometry of $\mathbf{R}^3$ that leaves a convex polyhedron P invariant carries a vertex to a vertex, an edge to an edge, and a face to a face. By definition, a *symmetry* of P is a *direct* isometry of $\mathbf{R}^3$ that leaves P invariant. By Theorem 1.1.1, any symmetry of our Platonic solid P must be a rotation. It also follows that if R is a symmetry rotation of P then the axis of rotation of R passes through a vertex, the midpoint of an edge, or the centroid of a face of P. The set of all symmetries of P forms a group, the *symmetry group* of P. Thus the symmetry group is a finite group of rotations. (In fact, it is a permutation group acting on the set of vertices.) The set of *all* isometries of $\mathbf{R}^3$ (direct or opposite) that leave P invariant is called the *extended (symmetry) group* of P. The extended group is still finite. In fact, since the product of two opposite isometries is direct, the extended group contains the symmetry group of P as a subgroup of index *at most* 2. In particular, the symmetry group is always normal in the extended group.

Let v_i, $i = 1, \ldots, V$, denote the vertices of a convex polyhedron P. We define the *center of mass* O by

$$O = \frac{1}{V} \sum_{i=1}^{V} v_i.$$

By convexity, O is contained in P. Moreover, O is left fixed by any isometry S of $\mathbf{R}^3$ that leaves P invariant. Indeed, since S carries vertices into vertices, it acts as a permutation on the set of vertices of P, and thereby S fixes O.

(Compare this with Problem 1.4.) In particular, O is a fixed point of the symmetry group of P.

Remark. For $V \geq 4$, the center of mass of the set of vertices of P is, in general, *different* from the *centroid* of P (considered as a solid of uniform mass density) as defined in integral calculus. (For details, see Berger [1], Section 2.7.5.) For P regular (defined below), the center of mass and the centroid coincide. In what follows, we will use the terminology "center of mass of P" meaning the center of mass of the set of vertices of P. For regular polyhedra we will also use the term centroid.

A *flag* in P is a triple (v, e, f), where v is a vertex, e is an edge, and f is a face of P such that $v \in e \subset f$. The polyhedron P is said to be *regular* if any two flags in P can be carried into each other by a suitable symmetry of P. Let P be regular. Then, by definition, the faces of P are regular polygons all congruent to each other, and the number of faces meeting at any vertex is the same. The *Schläfli-symbol* of a regular polyhedron P is the pair $\{a, b\}$, where the faces of P are regular a-sided polygons, and at each vertex b faces meet. *Double counting* the edges from vertex to vertex, and from face to face, we obtain

$$bV = 2E = aF. \tag{1.1.2}$$

Now a short computation in the use of (1.1.1) and (1.1.2) gives

$$V = \frac{4a}{2a + 2b - ab}, \quad E = \frac{2ab}{2a + 2b - ab}, \quad F = \frac{4b}{2a + 2b - ab}.$$

The common denominator must be positive: $2a + 2b - ab > 0$. Equivalently

$$(a - 2)(b - 2) < 4.$$

Since $a \geq 3$ (the faces of P have at least 3 sides) and $b \geq 3$ (at each vertex of P at least 3 edges meet), the *possible* Schläfli-symbols are

$$\{3, 3\}, \{3, 4\}, \{4, 3\}, \{3, 5\}, \{5, 3\}.$$

These are the Schläfli symbols of the (regular) tetrahedron, the octahedron, the cube, the icosahedron, and the dodecahedron. Since Plato gave them a prominent role in his Theory of Ideas, they are called the *Platonic solids.* The fact that the tetrahedron, octahedron, and the cube are regular in the sense above is not hard to show. We will discuss the construction and regularity of the icosahedron and the dodecahedron below.

Let v be a vertex of a Platonic solid P with Schläfli-symbol $\{a, b\}$. The *vertex figure* corresponding to v is the regular b-sided polygon whose vertices are the midpoints of the edges emanating from v. Regularity and in fact the very existence of the vertex figure (as a plane polygon) follow from regularity of P. Indeed, consider the symmetries of P that carry the flags (v, e, f) into each other, where e runs through all the edges of P emanating from v, and f runs through all faces of P meeting at v. These rotational

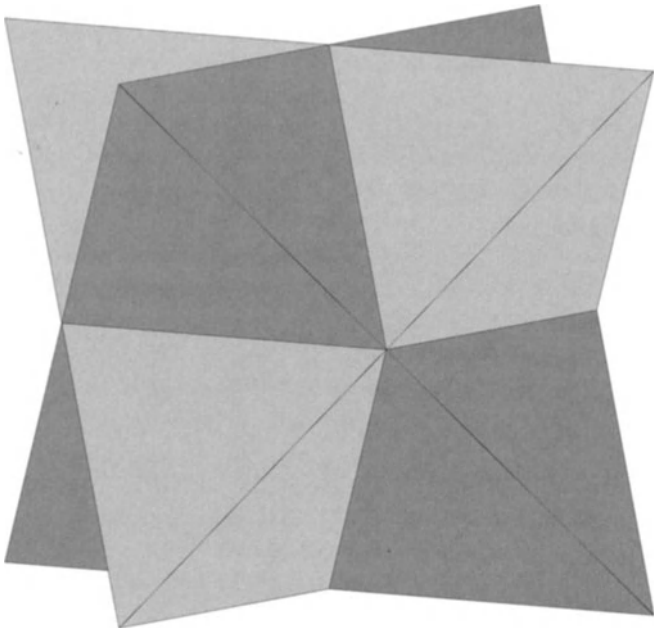


Figure 3.

symmetries have common axis of rotation ℓ passing through v, and therefore they form a cyclic group of order b. This group also permutes the vertices of the vertex figure cyclically. Thus the vertex figure is a regular (plane) polygon. The axis of rotation ℓ passes through the centroid of the vertex figure perpendicularly. The planes spanned by the vertex figures of P enclose a convex polyhedron P^0. It is clear that P^0 is regular since the symmetries of P also carry the vertex figures into one another. This also shows that the symmetry groups of P and P^0 are the *same*. By construction, the Schläfli-symbol of P^0 is $\{b, a\}$. We call P^0 the *dual* of P. Clearly $(P^0)^0 = P$. Figures 3, 4, and 5 depict a pair of dual tetrahedra, a dual pair of a cube and an octahedron, and a dual pair of an icosahedron and a dodecahedron. Notice that each edge of P bisects an edge of P^0 perpendicularly. By construction, there is a 1:1 correspondence between the vertices of P and the faces of P^0, and between the faces of P and the vertices of P^0.

Coming back to the question of existence and regularity, we see that dual pairs such as the icosahedron and dodecahedron coexist. Thus, it remains to show that, for example, the dodecahedron exists as a regular polyhedron. In order to accomplish that we first describe the "roof proof" for the existence of the dodecahedron. This is essentially contained in Book XIII of Euclid's Elements (Heath [1]).

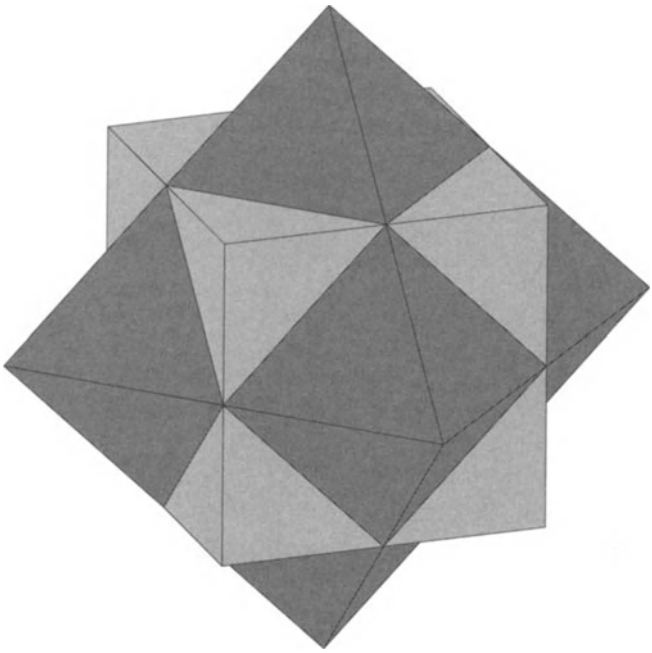


Figure 4.

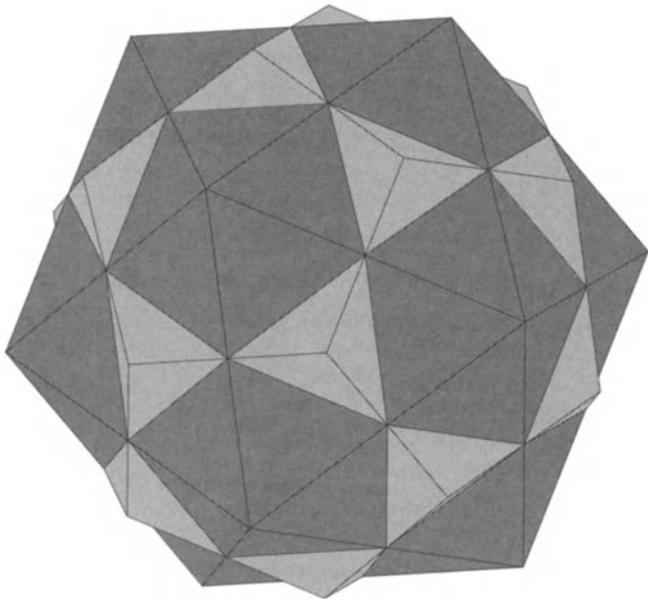


Figure 5.

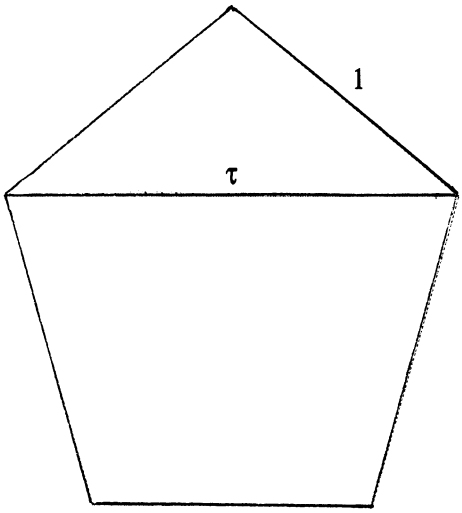


Figure 6.

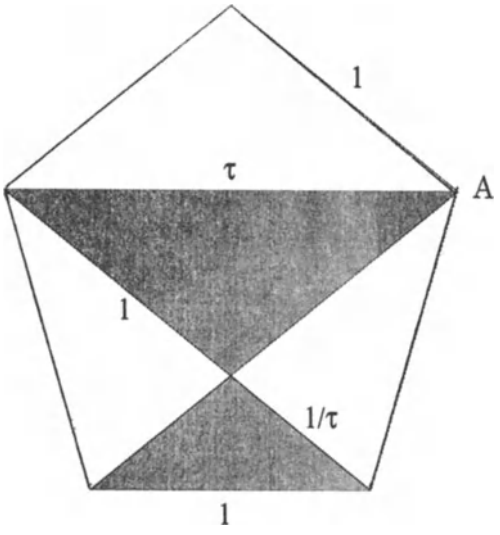


Figure 7.

A diagonal splits a regular pentagon with unit side length into an isosceles triangle and a symmetric trapezoid whose base length is, by definition, the golden section (Figure 6). Notice that similarity of the two isosceles triangles in Figure 7 gives the defining equality $\tau = 1 + 1/\tau$ of the golden section. Multiplying out, we obtain $\tau^2 = \tau + 1$, and the quadratic formula gives $\tau = (1 + \sqrt{5})/2$.



Figure 8.

We define a convex polyhedron with base square of side length τ, and let it have four additional faces: two isosceles triangles and two symmetric trapezoids, attached to each other along their unit length sides in an alternating manner. We call this polyhedron a *roof*. The top unit length side of the trapezoids forming a single edge (opposite to the base) is called the *ridge*. If it exists, the (solid) dodecahedron must be the union of an inscribed cube and six roofs. The bases of the roofs are the faces of the cube (see Figure 8).

To prove that the dodecahedron exists, we thus need to show that a triangular face and a trapezoidal face meeting at a common edge of the inscribed cube are coplanar (Figure 9). This amounts to showing that, in a single roof, the dihedral angle between a triangular face and the base and the dihedral angle between a trapezoidal face and the base are complementary. For this, we first take slices of the roof perpendicular to the base as in Figure 10, and use the Pythagorean theorem twice to conclude that the height of the roof is

$$\sqrt{1-(\tau/2)^2-((\tau-1)/2)^2}=1/2.$$

Similarity of the triangles in Figure 9 amounts to

$$\frac{\tau/2}{1/2}=\frac{1/2}{(\tau-1)/2}.$$

Multiplying out, this becomes $\tau(\tau-1)=1$, an equivalent form of the defining equality of the golden section. This completes the roof proof.

Checking regularity of the dodecahedron just constructed, though tedious, is now straightforward. Notice that the explicit enumeration of the symmetries acting on various flags is considerably simplified by relating the symmetries of the inscribed cube and the symmetries of the surrounding dodecahedron. As noted above, existence and regularity of the icosahedron follows by duality.

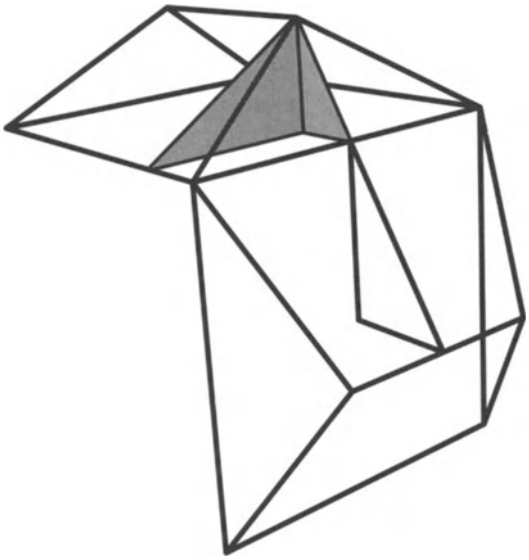


Figure 9.

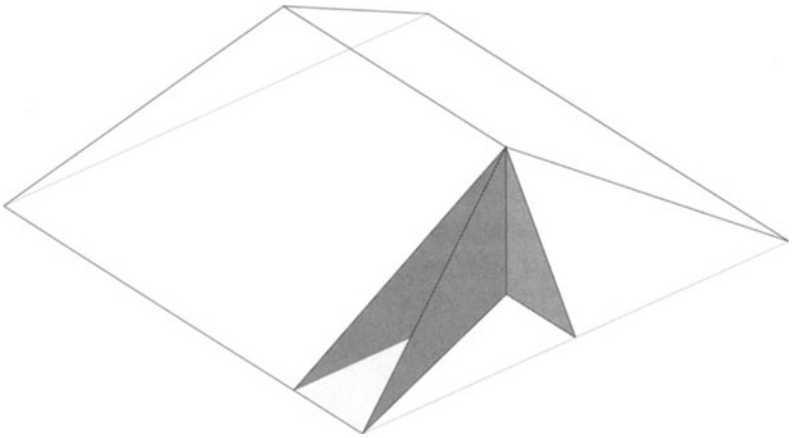


Figure 10.

We now briefly describe the symmetry group of each Platonic solid. We will occasionally make use of the extended group of Platonic solids.

Let P be a Platonic solid with Schläfli symbol $\{a, b\}$. By regularity, the symmetries of P are rotations whose axes pass through every vertex, the midpoint of every edge, and the centroid of every face. We claim that the symmetry group of P has order $2E$, where E is the number of edges of P. Indeed, there are $b-1$ (nontrivial) rotations with common axis of rotation passing through a vertex. There is exactly 1 rotation (actually a half-turn)

with axis of rotation passing through the midpoint of an edge. Finally, there are $a-1$ (nontrivial) rotations with common axis of rotation passing through the centroid of a face. With this we double count the number of rotations. Adding the identity, we see that the order of the symmetry group of P is

$$\frac{1}{2}(V(b-1)+E+F(a-1))+1.$$

Using (1.1.1)-(1.1.2), we rewrite this as

$$\frac{1}{2}(2E-V+E+2E-F)+1=\frac{1}{2}(5E-V-F)+1=\frac{1}{2}(4E-2)+1=2E.$$

The claim follows.

Since all Platonic solids have mirror symmetries (they are invariant under some reflections), for a Platonic solid, the symmetry group is an index 2 subgroup in the extended group.

Remark. Theorem 1.1.1 implies that every symmetry of a Platonic solid is a rotation. In contrast, not all opposite isometries that leave a Platonic solid invariant are reflections. For example, a regular tetrahedron is left invariant by rotatory reflections, rotations followed by reflections.

Returning to the main line, the observation that the symmetry (and extended) groups of a Platonic solid and its dual are the *same* reduces the cases to be considered to three: the tetrahedral, octahedral, and icosahedral groups.

A symmetry plane of the tetrahedron contains an edge and the midpoint of the opposite edge. Reflection to the symmetry plane interchanges two vertices and fixes the other two vertices, and the reflection is uniquely determined by this permutation on the 4 vertices (see Figure 11). This permutation is actually a *transposition* (a permutation that interchanges two letters while leaving all other letters fixed). It is a well-known fact in algebra that every permutation can be written as a product of transpositions. Thus, the extended group of the tetrahedron contains *all* permutations on the vertices, and this extended group is thereby isomorphic with the symmetric group $\mathcal{S}_4$ on 4 letters. Compositions of *even number of reflections* give the symmetry rotations of the tetrahedron (Problem 1.5), while compositions of even number of transpositions give the alternating subgroup $\mathcal{A}_4 \subset \mathcal{S}_4$. (The decomposition of a permutation into a product of transpositions is not unique, but the parity of the number (even-odd) of the participating transpositions in a product is unique.) We obtain that the *tetrahedral group*, the symmetry group of the tetrahedron, is isomorphic with the alternating subgroup $\mathcal{A}_4 \subset \mathcal{S}_4$ of even permutations on 4 letters.

The octahedron is a tetrahedron truncated along its vertex figures. Equivalently, the octahedron is the intersection of the tetrahedron and its dual (Figure 12). The symmetries of the tetrahedron thus become symmetries of the octahedron. In fact, the tetrahedral group must be of index 2 in

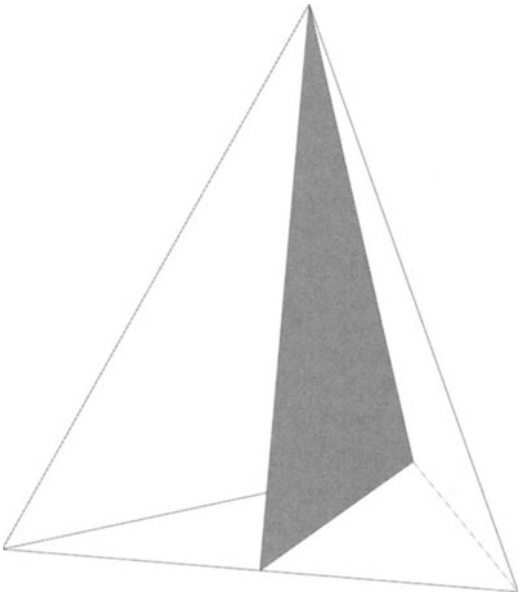


Figure 11.

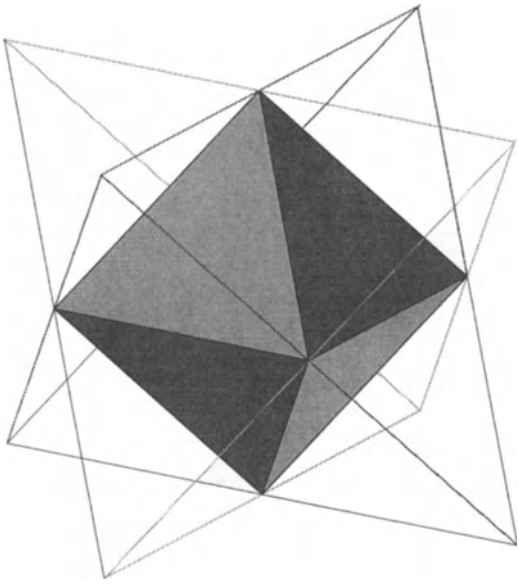


Figure 12.

the *octahedral group* since a symmetry of the octahedron either leaves the tetrahedron invariant or interchanges the tetrahedron and its dual. A more colorful picture emerges if we color the two tetrahedra in two different colors, let the octahedral faces inherit the colors, and consider color preserving and color reversing symmetries. From our study of the tetrahedral group we know that the color preserving symmetries form a group isomorphic with $\mathcal{A}_4$. There are 12 color-reversing symmetries: 6 half-turns with axes passing through the midpoint of 6 pairs of opposite edges of the octahedron, and 3 quarter-turns and their inverses with axes passing through 3 pairs of opposite vertices. It follows that the octahedral group is isomorphic with $\mathcal{S}_4$ (Problem 1.10(b)). (The 6 half-turns correspond to the 6 transpositions in $\mathcal{S}_4$, and the 3 quarter-turns and their inverses correspond to the 6 4-cycles in $\mathcal{S}_4$.)

Finally we turn to the case of the dodecahedron. We claim that the symmetry group of the dodecahedron is isomorphic with the alternating group $\mathcal{A}_5$ on 5 letters. This can be seen as follows:

First of all, recall from our "roof proof" that a cube can be inscribed in the dodecahedron whose 12 edges are diagonals of each of the 12 pentagonal faces of the dodecahedron. We call this cube *golden* since the ratio of the edges of the cube and the dodecahedron is the golden section. As Kepler observed, exactly 5 golden cubes can be inscribed in a dodecahedron. This is clear once we realize that on each face of the dodecahedron the 5 edges of the 5 golden cubes form the pentagram star. (To get a feel for how these cubes fit together, see Problem 1.11.)

Any symmetry of the dodecahedron must carry a golden cube to a golden cube, and therefore the symmetries act as permutations on the 5 golden cubes. This representation of the symmetry group of the dodecahedron as a permutation group on the set of 5 golden cubes is *faithful* in the sense that no nontrivial rotational symmetry of the dodecahedron acts as the identity permutation on the cubes. (Indeed, the rotation axis of a symmetry passes through a vertex, the midpoint of an edge, or the centroid of a face of the dodecahedron. In each case, faithfulness follows easily.) Since the dodecahedron has 30 edges, its symmetry group has 60 elements, the same number as the order of $\mathcal{A}_5$. We conclude that the permutation group that represents the symmetries of the dodecahedron, and the alternating group $\mathcal{A}_5$ are both index 2 (thereby normal) subgroups in the symmetric group $\mathcal{S}_5$ on 5 letters. On the other hand, $\mathcal{A}_5$ is the *only* index 2 subgroup in $\mathcal{S}_5$. This follows from the well-known fact that $\mathcal{A}_5$ is *simple* (no proper normal subgroup) by an elementary argument. Hence our permutation group must be equal to $\mathcal{A}_5$, and the claim follows.

For the final step, one can also give an alternative geometric argument, and prove first that the symmetry group of the dodecahedron is simple (Problem 1.12), therefore the permutation group representing the symmetries of the dodecahedron is also simple; and finally conclude that this permutation group must coincide with $\mathcal{A}_5$.

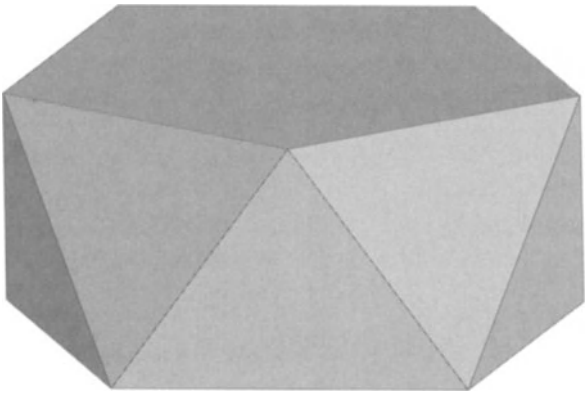


Figure 13.

Remark. The extended group of the dodecahedron does not act faithfully on the 5 golden cubes. Indeed, reflection in the centroid is an opposite isometry that leaves the dodecahedron and each of the 5 golden cubes invariant.

The dodecahedron and the icosahedron are dual. In particular, the symmetry group of the icosahedron, the *icosahedral group*, is also isomorphic with $\mathcal{A}_5$. It will be important in what follows to obtain an explicit construction for the icosahedron, and an independent description of the icosahedral group. Let us think of the icosahedron as being made up of two pentagonal pyramids separated by a *pentagonal antiprism*, a belt of 10 equilateral triangular faces (Figure 13). Any *cross edge* in the antiprism (connecting the top and bottom pentagonal faces) has an opposite parallel cross edge. These two edges are the opposite edges of a rectangle which is *golden*, i.e. the ratio of two adjacent side lengths is τ. This is because the base of the rectangle is a diagonal of one of the pentagonal faces of the antiprism (Figure 14). Picking opposite edges consistently, we find that the 12 vertices of the icosahedron are on 3 mutually perpendicular golden rectangles (Figure 15). This model of the icosahedron is due to Pacioli, who, in his *De Divina Proportione*, called it the "twelfth almost incomprehensible effect". This medieval classic, richly illustrated with drawings of models made by Leonardo da Vinci, contains a collection on the occurrence of the golden number in a variety of (actually 13) geometric situations. Due to the systematic occurrence of the golden number, Kostant [1] describes the icosahedron "... as a symphony in the golden number."

Dual to Kepler's compound of the 5 cubes inscribed in a dodecahedron, 5 octahedra can be inscribed in the icosahedron. Indeed, the 15 opposite pairs of edges of the icosahedron give 15 golden rectangles inscribed in the icosahedron. By further grouping, these 15 golden rectangles form 5 configurations, each consisting of 3 mutually perpendicular golden rectangles.

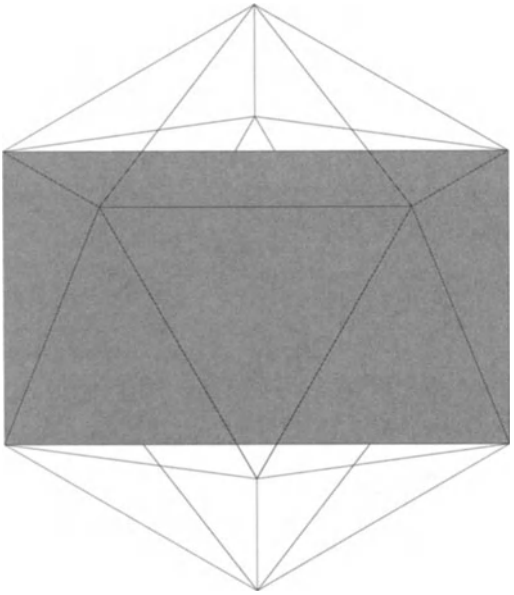


Figure 14.

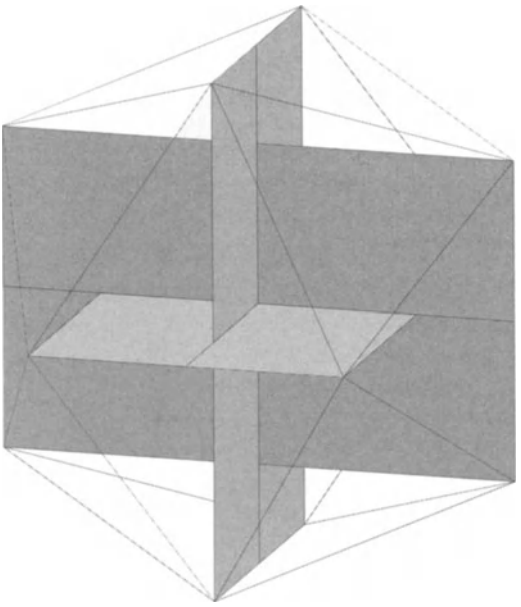


Figure 15.

One can associate an octahedron to each configuration: The 6 midpoints of the 6 shorter edges of the golden rectangles give the 6 vertices of the octahedron. Putting everything together, we arrive at the compound of 5 octahedra. By duality, the symmetry group $\mathcal{A}_5$ acts faithfully on the 5 octahedra inscribed in the icosahedron.

A more colorful picture is obtained if we observe that the faces of the icosahedron can be colored by 5 different colors such that the 4 faces in one color group are disjoint, i.e. they do not even touch at a vertex. (An algorithm for finding one color group is as follows: Imagine that you are standing on a face f with bounding edges e_i, $i = 1, 2, 3$. For each i, step on the face f_i adjacent to f across e_i. At the vertex v_i of f_i opposite to e_i, five faces meet. Only two of these faces are disjoint from f. These two faces appear to you to the right and to the left. If you are right-handed, add the right face to the color group of f, and if you are left-handed, add the left face to the color group of f.) (For a connection with the four-coloring of the vertices of the dodecahedron, see Problem 1.15.)

Extending the four faces in one color group we obtain a tetrahedron circumscribed around the icosahedron. By regularity of the icosahedron, this tetrahedron must be regular.

Since there are five color groups, we obtain five regular tetrahedra circumscribed around the icosahedron. The five tetrahedra corresponding to the five color groups (Figure 16) constitute a single orbit of an order 5 symmetry rotation R of the icosahedron, i.e. the five iterates of R carry the five tetrahedra into each other cyclically. This follows by observing first that at each vertex of the icosahedron the five faces are colored with five different colors. Thus, the order 5 rotation with axis passing through this vertex must be R. Thus, R must have angle $2\pi/5$, and, by the Pacioli model, the axis of rotation must contain a diagonal of a golden rectangle. (Problem 1.16 asks to determine the orthogonal matrix of R explicitly.) Note also that the 20 vertices of the 5 tetrahedra are the vertices of a regular polyhedron. Thus, it must be a dodecahedron. This circumscribed dodecahedron is homothetic to the dodecahedral dual (Figure 17).

We now return to the classification of finite isometry groups. Let G be a finite group of isometries of $\mathbf{R}^3$. We first claim that G fixes a point $O \in \mathbf{R}^3$. Indeed, consider the *orbit*

$$G(p) = \{R(p) \mid R \in G\}$$

passing through a point $p \in \mathbf{R}^3$. Since G is finite, we can write

$$G(p) = \{p_1, \ldots, p_n\}, \quad p = p_1 \in \mathbf{R}^3.$$

By definition, G acts on $G(p)$ by permuting the elements in the orbit. Hence the center of mass

$$O = \frac{1}{n} \sum_{i=1}^{n} p_i$$

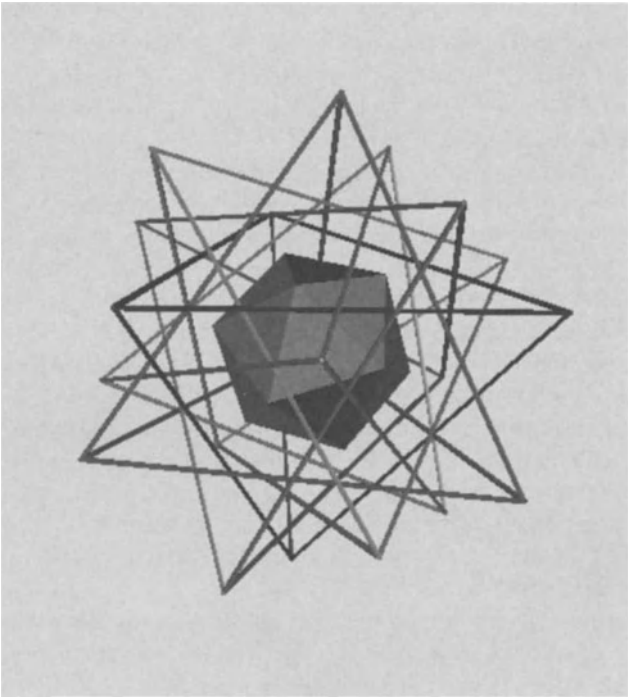


Figure 16.

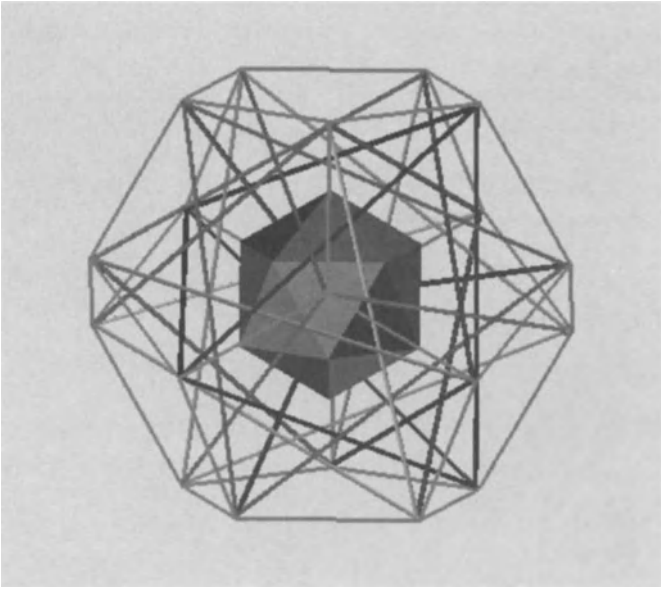


Figure 17.

of $G(p)$ is left fixed by all the elements of G. The claim follows. (For a different proof, see Problem 1.4.)

In the classification problem we first treat the case when G consists of *direct* isometries only. For simplicity, in the following argument we denote the unit sphere *with center at* O by S^2. By Theorem 1.1.1, each nontrivial element R in G is a rotation. The axis of rotation passes through O and intersects S^2 in two antipodal points. We call these points *poles*, and denote the set of poles for all the rotations in G by Π. Since G is finite, the *isotropy group*

$$G_x = \{R \in G \mid R(x) = x\}$$

at a pole $x \in \Pi$ is cyclic. In fact, taking the smallest positive angle rotation R_0 in G_x, any $R \in G_x$, having the same axis as R_0, must have angle of rotation a multiple of the smallest angle. Thus R is an iterate of R_0, and R_0 generates G_x. We call the order d of R_0 the *degree* of x. It is also the order of G_x.

If R is a rotation in G and S is *any* isometry, then the axis of the rotation of $S \circ R \circ S^{-1}$ is the S-image of the axis of rotation of R. Π is G-invariant. In particular, since

$$(S \circ R \circ S^{-1})^m = S \circ R^m \circ S^{-1}, \quad m \in \mathbf{Z},$$

the poles in a G-orbit of Π have the same degree. If $C \subset \Pi$ is a G-orbit, we denote this common degree by $d = d_C$. Since d_C is also the order of the isotropy group G_x, for any $x \in C$, we have

$$|C| = |G(x)| = \frac{|G|}{|G_x|} = \frac{|G|}{d_C},$$

where the middle equality is an elementary "counting formula" valid for any finite group action.

Now we count how many nontrivial rotations have poles in C. Each axis, giving two poles in C, is the axis of $d_C - 1$ nontrivial rotations. Thus, the number of nontrivial rotations with poles in C is

$$\frac{1}{2}(d_C - 1)|C| = \frac{1}{2}(d_C - 1)\frac{|G|}{d_C},$$

where the numerical factor $1/2$ is because each axis of rotation gives two poles. Thus the total number of nontrivial rotations in G is

$$|G| - 1 = \frac{1}{2}|G| \sum_{C \in \Pi/G} \frac{d_C - 1}{d_C}.$$

Here the summation runs through the orbits of poles. We write this as the diophantine equation

$$2 - \frac{2}{|G|} = \sum_{C \in \Pi/G} \left(1 - \frac{1}{d_C}\right). \tag{1.1.3}$$

We may assume that G is nontrivial: $|G| \geq 2$. Since

$$1 \leq 2 - \frac{2}{|G|} < 2$$

the number of terms in the summation (1.1.3) can only be 2 or 3.

Assume first that we have 2 orbits C_1, C_2, with $d_{C_1} = d_1$, $d_{C_2} = d_2$. Then (1.1.3) reduces to

$$2 - \frac{2}{|G|} = \left(1 - \frac{1}{d_1}\right) + \left(1 - \frac{1}{d_2}\right),$$

or equivalently,

$$\frac{|G|}{d_1} + \frac{|G|}{d_2} = 2.$$

On the left-hand side, the terms are positive integers since $|C_1| = |G|/d_1$ and $|C_2| = |G|/d_2$. Thus both terms must be equal to one. We obtain

$$d_1 = d_2 = |G|.$$

This means that G is cyclic and consists of rotations around a single axis intersecting S^2 in an antipodal pair of poles, and each orbit is a single point.

Assume now that there are 3 equivalence classes C_1, C_2, C_3, with $d_{C_1} = d_1$, $d_{C_2} = d_2$, $d_{C_3} = d_3$. Then (1.1.3) takes the form

$$2 - \frac{2}{|G|} = \left(1 - \frac{1}{d_1}\right) + \left(1 - \frac{1}{d_2}\right) + \left(1 - \frac{1}{d_3}\right).$$

We rewrite this as

$$\frac{1}{d_1} + \frac{1}{d_2} + \frac{1}{d_3} = 1 + \frac{2}{|G|}.$$

Since $1/3 + 1/3 + 1/3 = 1 < 1 + 2/|G|$ there must be at least one degree equal to 2. We may assume that it is d_3. The equality above reduces to

$$\frac{1}{d_1} + \frac{1}{d_2} = \frac{1}{2} + \frac{2}{|G|}.$$

A little algebra now shows

$$(d_1 - 2)(d_2 - 2) = 4(1 - d_1 d_2/|G|) < 4.$$

Letting $d_1 \leq d_2$, the possible values of d_1, d_2, d_3 and $|G|$ are given in the following table:

d_1	2	3	3	3
d_2	d	3	4	5
d_3	2	2	2	2
$\|G\|$	$2d$	12	24	60

The remaining step is to analyze each numerical column.

We first let $d_1 = 2, d_2 = d, d_3 = 2$, and $|G| = 2d$. We have 3 orbits in Π that contain $2d/2 = d$, $2d/d = 2$, and $2d/2 = d$ elements. An isotropy group of the orbit with two elements must be generated by a rotation R of order d. (Recall that any isotropy group is cyclic.) Thus the powers of R form a cyclic subgroup C_d of order d in G. We assume that $d \geq 3$. (For $d = 1$, G is clearly cyclic, and, for $d = 2$, a simple analysis shows that G is isomorphic with $C_2 \times C_2$.) Apart from the cyclic subgroup C_d, the rest of G is made up by d half-turns. The axis of R is perpendicular to the axes of the half-turns. This is because the two poles that correspond to R form a single orbit, and these two poles must be interchanged by each half-turn. Thus the $2d$ poles corresponding to the d half-turns are coplanar, and they are divided into two orbits consisting of d poles each. R is a self-map on each orbit of poles. The only way this can be possible is if the angle between adjacent axes of the d half-turns is π/d. We conclude that G is isomorphic with the *dihedral group* D_d, a group of order $2d$, made up by a cyclic subgroup C_d of order d, and d elements of order 2. (Since D_2 is isomorphic with $C_2 \times C_2$, G is also dihedral in the excluded case $d = 2$.) Applying an isometry of $\mathbf{R}^3$, the fixed point O can be brought to the origin. In addition, the axis of R can be brought to the third axis, and an axis of a half-turn can be brought to the first axis. The isometry that makes this configuration of axes conjugates G into the the standard form of D_d. With this, D_d can be thought of as the symmetry group of a regular *d-prism* of height h: $P_d \times [-h/2, h/2] \subset \mathbf{R}^2 \times \mathbf{R} = \mathbf{R}^3$, where P_d is the regular d-sided polygon inscribed in the unit circle $S^1 \subset \mathbf{R}^2$ with a vertex at $(1, 0)$. (For $d = 4$, we assume that $h \neq \sqrt{2}$ so that $P_d \times [-h/2, h/2]$ is not a cube.)

In a similar vein, the trivial case of a cyclic $G = C_d$ with two orbits can be viewed geometrically as the symmetry group of a *regular pyramid* with base P_d. (Again, for $d = 3$, the height of the pyramid should be $\neq \sqrt{2}$, since otherwise we end up with a regular tetrahedron.)

Next we will consider the second numerical column of the table. It is clear that the tetrahedral group furnishes an example for this numerical column. It is less clear, however, that there are no other groups which have this orbit structure. Our task is to show that this is in fact the case. To do this we need to construct a tetrahedron whose symmetry group is G. We have 3 orbits in Π that contain $12/3 = 4$, $12/3 = 4$, and $12/2 = 6$ elements. Let $x \in \Pi$ be a pole in an orbit containing 4 elements, and pick a $y \in \Pi$ different from x and its antipodal. Let R be a generator of the isotropy subgroup G_x. By our choice, R has order 3. The orbit $G_x(y)$ (of the isotropy group G_x passing through y) consists of the three vertices of an equilateral triangle that lies in a plane perpendicular to the rotation axis of R through x. The axis of R passes through the centroid of the triangle. Interchanging x and y, we see that x, y, $R(y)$, and $R^2(y)$ are the vertices of a regular tetrahedron. Since Π is G-invariant, the symmetry group of the

tetrahedron is G. (Notice that if we pick the other orbit of 4 elements at the start, we end up with the dual tetrahedron.)

We now consider the third numerical column. This time, we note that this case should correspond to the octahedral group. We have three orbits in Π that contain $24/3 = 8$, $24/4 = 6$, and $24/2 = 12$ orbits. The three orbits contain different number of elements, in particular, each orbit must be invariant under the antipodal map of S^2. Let $x \in \Pi$ be a pole in the orbit containing 6 elements, and pick $y \in \Pi$ different from x and its antipodal. Let R be a generator of G_x. Since G_x has four elements, R is a quarter-turn, and the orbit $G_x(y)$ consists of the vertices of a square lying in the perpendicular bisector of the segment connecting x and its antipodal. Interchanging the roles of x and y, we see that $G_x(y)$ along with x and its antipodal form the vertices of an octahedron with symmetry group G.

Finally, we consider the fourth numerical column. This case should correspond to the icosahedral group. We have three orbits in Π that contain $60/3 = 20$, $60/5 = 12$, and $60/2 = 30$ elements. Once again, each orbit is invariant under the antipodal map. Let $x \in \Pi$ be a pole in the orbit containing twelve elements, and pick $y \in \Pi$ *closest to* x. Let R be a generator of the isotropy group G_x of order 5. The orbit $G_x(y)$ gives the vertices of a regular pentagon lying in a plane perpendicular to the rotation axis of R passing through x. This plane *cannot be* the perpendicular bisector of the segment connecting x and its antipodal (since otherwise Π would contain only $1 + 5 + 1 = 7$ elements). Thus the six elements: x, $R^j(y)$, $j = 0, \ldots, 4$, and their antipodals are all distinct, and so they must comprise the (twelve-element) G-orbit through x. A mildly tedious argument shows that this orbit forms the vertices of an icosahedron with symmetry group G.

We obtain the following result of Klein [1]:

Theorem 1.1.2. *The only finite groups of direct isometries of* $\mathbf{R}^3$ *are the cyclic groups* C_d, *the dihedral groups* D_d, *the tetrahedral group* $\mathcal{A}_4$, *the octahedral group* $\mathcal{S}_4$, *and the icosahedral group* $\mathcal{A}_5$.

Remark. For a different "grungy classification" of the finite groups of direct isometries of $\mathbf{R}^3$, see Thurston [1], p. 245.

We conclude this section by allowing opposite isometries in our group G. Let G be any finite group of isometries of $\mathbf{R}^3$. By the center-of-mass argument above, G has a fixed point O. Let $G^+ \subset G$ denote the subgroup consisting of the direct isometries of G. The possible choices of G^+ are listed in Theorem 1.1.2. To get something new, we may assume that $G^+ \subset G$ is a *proper* subgroup. Since the composition of two opposite isometries is direct, G^+ is of index 2 in G. In other words, G^+ and any element in $G^- = G - G^+$ generate G. Thus, we have to study the possible configurations of G^+ listed in Theorem 1.1.2, and a single opposite isometry.

Let $A : \mathbf{R}^3 \to \mathbf{R}^3$ be the reflection in the fixed point O given by $p+A(p) = 2O$. (Restricted to S^2, A is the antipodal map.) Note that A is an order 2 opposite isometry of $\mathbf{R}^3$ that commutes with any isometry that fixes O. In particular, A commutes with (the elements in) G. A may or may not belong to G.

If $A \in G$ then $G^- = A \cdot G^+$ since the right-hand side is a set of $|G^+|$ opposite isometries in G. Since A commutes with G, the cyclic subgroup $\{I, A\} \subset G$ (isomorphic with C_2) is normal. Since index 2 subgroups (such as $G^+ \subset G$) are always normal, we obtain the isomorphism $G \cong G^+ \times C_2$. Using Theorem 1.1.2, we arrive at the list

$$C_d \times C_2,\ D_d \times C_2,\ \mathcal{A}_4 \times C_2,\ \mathcal{S}_4 \times C_2,\ \mathcal{A}_5 \times C_2. \tag{1.1.4}$$

Assume now that $A \notin G$. We first show that G^+ is contained in a (finite) group G^* of *direct* isometries as an index 2 subgroup. To do this, we define

$$G^* = G^+ \cup A \cdot G^-.$$

Notice that G^* is a group. (Since A commutes with G the set G^* is closed under composition and inverse.) Moreover, G^* consists of direct isometries since A and the elements of G^- are opposite. G^+ and $A \cdot G^-$ are disjoint since $A \notin G$. In particular, $|G^*| = 2|G^+|$, and G^+ is an index 2 subgroup in G^*. The possible inclusions $G^+ \subset G^*$ are easily listed since all the isometries involved are direct and Theorem 1.1.2 applies. We obtain

$$C_d \subset C_{2d},\ C_d \subset D_d,\ D_d \subset D_{2d},\ \mathcal{A}_4 \subset \mathcal{S}_4.$$

Finally, G can be recovered from G^* via the formula

$$G = G^+ \cup A \cdot (G^* - G^+).$$

In general, we denote by G^*G^+ the group defined by the right-hand side of this equality, when $G^+ \subset G^*$ is an inclusion of finite groups of direct isometries, and G^+ is of index 2 in G^*. We finally arrive at the list

$$C_{2d} \cdot C_d,\ D_d \cdot C_d,\ D_{2d} \cdot D_d,\ \mathcal{S}_4 \cdot \mathcal{A}_4. \tag{1.1.5}$$

Theorem 1.1.3. *The finite groups of isometries of* $\mathbf{R}^3$ *containing an opposite isometry are listed in (1.1.4) and (1.1.5).*

Example 1.1.4. The octahedron and the icosahedron are centrally symmetric (reflection in the centroid is a symmetry), while the tetrahedron is not. Thus, the extended group of the tetrahedron is $\mathcal{S}_4 \cdot \mathcal{A}_4$, the extended group of the octahedron is $\mathcal{S}_4 \times C_2$, and the extended group of the icosahedron is $\mathcal{A}_5 \times C_2$.

As noted in the proof of Theorem 1.1.1, translating the fixed point of a finite group G of isometries of $\mathbf{R}^3$ to the origin, (a conjugate of) G becomes linear. The group of all linear isometries of $\mathbf{R}^3$ is the *orthogonal group* $O(3)$, the group of all orthogonal 3×3-matrices. A linear isometry

of $\mathbf{R}^3$ is direct or opposite according to whether its determinant is $+1$ or -1. The group of all direct linear isometries of $\mathbf{R}^3$ is the *special orthogonal group* $SO(3)$, the subgroup of $O(3)$ consisting of orthogonal 3×3-matrices with determinant 1.

Theorem 1.1.1 asserts that an element of $SO(3)$ can be viewed as a rotation with rotation axis passing through the origin. Theorems 1.1.2-1.1.3 classify all finite subgroups of $SO(3)$ and $O(3)$.

1.2 Rotations and Möbius Transformations

Our main purpose in the next two sections is to classify all finite Möbius groups, finite groups of linear fractional transformations under composition. In this section we will assume the knowledge of some basic material in complex analysis, notably conformal transformations and holomorphic branched coverings. The main references are Ahlfors [1] and Farkas-Kra [1], Chapter 1.

Recall that a *linear fractional transformation* is a transformation of the *extended complex plane* $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ of the form

$$\zeta \mapsto \frac{a\zeta + b}{c\zeta + d}, \quad \zeta \in \hat{\mathbf{C}}, \ ad - bc = 1, \ a, b, c, d \in \mathbf{C}. \tag{1.2.1}$$

(The mapping properties at ∞ are understood in the limiting sense, i.e. ∞ is sent to a/c, and $\zeta = -d/c$ is sent to ∞.) It is an elementary computation to check that the linear fractional transformations form a group under composition and inverse (as transformations). (See Problem 1.20.) In fact, it is well-known that the linear fractional transformations are precisely the conformal transformations of $\hat{\mathbf{C}}$.

We call the group of linear fractional transformations the *Möbius group*, and denote it by $\mathcal{M}(\hat{\mathbf{C}})$.

We will be primarily interested in the (compact) subgroup $\mathcal{M}_0(\hat{\mathbf{C}}) \subset \mathcal{M}(\hat{\mathbf{C}})$ consisting of linear fractional transformations of the form

$$\zeta \mapsto \frac{z\zeta - \bar{w}}{w\zeta + \bar{z}}, \quad \zeta \in \hat{\mathbf{C}}, \ |z|^2 + |w|^2 = 1, \ z, w \in \mathbf{C}. \tag{1.2.2}$$

Let $SL(2, \mathbf{C})$ denote the *special linear group*. By definition, an element of $SL(2, \mathbf{C})$ is a complex 2×2-matrix with determinant 1:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad ad - bc = 1, \ a, b, c, d \in \mathbf{C}. \tag{1.2.3}$$

Comparing this with (1.2.1), we say that (1.2.3) *defines* the linear fractional transformation (1.2.1). As simple computation shows, associating to the matrix (1.2.3) the linear fractional transformation (1.2.1) gives rise to a homomorphism $\pi : SL(2, \mathbf{C}) \to \mathcal{M}(\hat{\mathbf{C}})$ onto the Möbius group. (See Problem 1.21.)

In (1.2.1), a, b, c, d and their negatives give the same linear fractional transformation. Conversely, the linear fractional transformation (1.2.1) determines a, b, c, d up to a common sign. Thus, π is $2:1$ with kernel $\{\pm I\}$. In particular, we have the isomorphism

$$\mathcal{M}(\hat{\mathbf{C}}) \cong SL(2, \mathbf{C})/\{\pm I\}.$$

Under π, $\mathcal{M}_0(\hat{\mathbf{C}})$ is twofold covered by the subgroup $SU(2) \subset SL(2, \mathbf{C})$ of *special unitary matrices* with typical element

$$\begin{bmatrix} z & -\bar{w} \\ w & \bar{z} \end{bmatrix}, \quad |z|^2 + |w|^2 = 1, \ z, w \in \mathbf{C}. \tag{1.2.4}$$

Once again, (the restriction of) π establishes the isomorphism

$$\mathcal{M}_0(\hat{\mathbf{C}}) \cong SU(2)/\{\pm I\}.$$

Let h denote the *stereographic projection* of the 2-sphere $S^2 \subset \mathbf{R}^3$ onto $\hat{\mathbf{C}}$. Under h, the north pole $N = (0, 0, 1) \in S^2$ corresponds to $\infty \in \hat{\mathbf{C}}$. Given $x \in S^2$, $x \neq N$, $h(x) \in \mathbf{C}$ is defined by the requirement that the three points N, x and $h(x)$ are collinear. Analytically

$$h(x) = \frac{x_0 + ix_1}{1 - x_2}, \quad x = (x_0, x_1, x_2) \in S^2, \quad x_2 \neq 1. \tag{1.2.5}$$

The inverse of h is given by

$$h^{-1}(\zeta) = \left(\frac{2\zeta}{|\zeta|^2 + 1}, \frac{|\zeta|^2 - 1}{|\zeta|^2 + 1} \right) \in S^2, \quad \zeta \in \mathbf{C}, \tag{1.2.6}$$

where $\mathbf{C}$ is imbedded in $\mathbf{R}^3$ as the coordinate plane spanned by the first two coordinate axes. (See Problem 1.22.)

It is well-known that h establishes a conformal equivalence between S^2 and $\hat{\mathbf{C}}$.

For the discussion that follows it is important to introduce a specific notation for *linear* rotations. Given $x \in S^2$ and $\theta \in \mathbf{R}$, let $R_{\theta,x}$ denote the linear rotation with angle θ and axis $\mathbf{R} \cdot x$. (Notice that $\pm x$ are the poles of $R_{\theta,x}$ as defined in the previous section.)

By Theorem 1.1.1, the rotation $R_{\theta,x}$ is a typical element of $SO(3)$.

Clearly, $R_{-\theta,-x} = R_{\theta,x}$. Moreover, for $x, y \in S^2$ and $\theta, \phi \in \mathbf{R}$, we have

$$R_{\phi,y} \circ R_{\theta,x} \circ R_{\phi,y}^{-1} = R_{\theta, R_{\phi,y}(x)}. \tag{1.2.7}$$

Indeed, the left-hand side is a rotation (Theorem 1.1.1) with axis $R_{\phi,y}(x)$ (the fixed point set), and conjugation does not change the angle of rotation.

The following theorem is due to Cayley in 1879:

Theorem 1.2.1. *Let R be the rotation $R_{\theta,x}$, $x \in S^2$, $\theta \in \mathbf{R}$, restricted to S^2. Then R conjugated with the stereographic projection h is the linear fractional transformation*

$$(h \circ R \circ h^{-1})(\zeta) = \frac{z\zeta - \bar{w}}{w\zeta + \bar{z}}, \quad \zeta \in \hat{\mathbf{C}}, \tag{1.2.8}$$

where

$$z = \cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)x_2 \text{ and } w = \sin\left(\frac{\theta}{2}\right)(x_1 + ix_0). \tag{1.2.9}$$

PROOF. First of all, for $x = N$, (1.2.8)-(1.2.9) reduce to

$$(h \circ R \circ h^{-1})(\zeta) = e^{i\theta}\zeta,$$

and the theorem follows in this case. In what follows, we may assume that $x \neq N$.

In general, $h \circ R \circ h^{-1}$ on the left-hand side of (1.2.8) is the composition of an isometry and two conformal equivalences. Therefore it is a conformal transformation of $\hat{\mathbf{C}}$. Hence, it must be a linear fractional transformation. It remains to show that this linear fractional transformation has the specific form given by the right-hand side of (1.2.8) with coefficients in (1.2.9). To prove this we use the elementary fact that a linear fractional transformation is uniquely determined by its action on 3 points. (A linear fractional transformation is determined by 4 complex parameters $a, b, c, d \in \mathbf{C}$ with one complex condition $ad - bc = 1$.) For the first two points we choose the fixed points of the transformation defined by the right-hand side of (1.2.8). To obtain these fixed points, we rewrite

$$\frac{z\zeta - \bar{w}}{w\zeta + \bar{z}} = \zeta$$

as the quadratic equation

$$w\zeta^2 - 2i\Im(z)\zeta + \bar{w} = 0.$$

Solving for ζ and using (1.2.9), we find that the fixed points are

$$\frac{x_2 \pm 1}{x_0 - ix_1}.$$

By (1.2.5) and $|x|^2 = 1$, we see that these two points coincide with $h(\pm x)$, the fixed points of $h \circ R \circ h^{-1}$. (Compare the direct computation with Problem 1.25.) For the third point we choose $\zeta = \infty = h(N)$. Substituting this into (1.2.8), we need to verify that $h(R(N)) = z/w$, where z and w are given in (1.2.9). Indeed, since $|z|^2 + |w|^2 = 1$, by (1.2.6), we have

$$h^{-1}(z/w) = (2z\bar{w}, |z|^2 - |w|^2) \in S^2. \tag{1.2.10}$$

A bit of analytic geometry now shows that this is $R(N) = R_{\theta,x}(N)$. The theorem follows.

Remark. (1.2.10) gives the components of the Hopf map which is treated in Section 1.4.

To simplify the terminology, we say that the linear fractional transformation (1.2.8)-(1.2.9) and the rotation $R_{\theta,x}$ *correspond to each other.*

Corollary 1.2.2. *We have the isomorphisms*

$$SO(3) \cong \mathcal{M}_0(\hat{\mathbf{C}}) = SU(2)/\{\pm I\}.$$

Recall that our goal is to list all finite Möbius groups $G \subset \mathcal{M}(\hat{\mathbf{C}})$. In this case we do not have as much Euclidean structure as for finite rotation groups (classified in the previous section), but a simple observation gives a clue as to how to proceed:

Assume that we can find a rational function $q = q_G : \mathbf{C} \to \mathbf{C}$ whose *invariance group* is G, i.e. for $g \in \mathcal{M}(\hat{\mathbf{C}})$, $q \circ g = q$, iff $g \in G$. Being rational, q can be extended to a holomorphic self-map of $\hat{\mathbf{C}}$ (denoted by the same symbol). Since G is the invariance group of the rational q, the extension is a holomorphic $|G|$-fold branched covering $q : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$.

In general, a nonconstant holomorphic map $f : M \to N$ between compact Riemann surfaces is an *n-fold branched covering* with finitely many *branch points* in M. Near a branch point on M and near its f-image (called a *branch value*), local coordinates ζ and ξ can be introduced that vanish at the branch point ($\zeta = 0$) and at the branch value ($\xi = 0$) such that, in these coordinates, f has the form $\xi = \zeta^m$. We call $m-1$ the *branch number* of f at the branch point.

In our case, the branch points of the extended q are the fixed points of the linear fractional transformations in G, and the branch numbers correspond to the orders of the rotations (minus 1) *in the special case when G is defined by a rotation group* via Theorem 1.2.1. By the Riemann-Hurwitz formula (Problem 1.27), the total branch number (the sum of all branch numbers) is $2|G| - 2$. This will give the diophantine restriction (1.1.3) for G. It is now clear that our main purpose should be to construct rational functions invariant under finite Möbius groups. To do this, we first exhibit a list of finite subgroups of $\mathcal{M}_0(\hat{\mathbf{C}})$. Let G be one of the groups in this list. G can be "linearized" by lifting it to $SU(2)$ along the 2:1 projection $\pi : SU(2) \to \mathcal{M}_0(\hat{\mathbf{C}})$. We call this the *binary group associated to G* and denote it by G^*.

We then solve the invariance problem for homogeneous polynomials in two variables for G^*, that is, we find all homogeneous polynomials $\xi : \mathbf{C}^2 \to \mathbf{C}$ with invariance group G^*. By homogeneity, quotients of some of these invariant polynomials define holomorphic self-maps of the complex projective line $\mathbf{C}P^1$. On the other hand, since $\mathbf{C}P^1 = \hat{\mathbf{C}}$, these quotients also restrict to rational functions (on $\mathbf{C}$), and, as it turns out, they have invariance group G.

Finally, to prove that *any* finite Möbius group G is conjugate to one of the finite subgroups of $\mathcal{M}_0(\hat{\mathbf{C}})$ in our list, we construct a rational function with invariance group G, and compare it with the rational functions in the list we constructed previously. This amounts to comparing singularities of rational functions à la Riemann, a well-understood method in complex analysis. For brevity, we will do this last step by appealing to uniformization.

Remark. There is a quick algebraic way to find all finite Möbius groups G. It is based on the fact that any finite subgroup of $SL(2, \mathbf{C})$ (such as the binary cover G^* of G) is conjugate to a subgroup of $SU(2)$ (and the finite subgroups of $SU(2)$ can be classified easily). Conjugacy is usually proved by averaging the standard scalar product on $\mathbf{C}^2$ over the finite subgroup of $SL(2, \mathbf{C})$. (For details, see Schurman [1].) We prefer to follow a longer path here because we will use some of the ingredients later.

We start our list with the cyclic group C_d of order d. It is realized as the group of rotations of $\hat{\mathbf{C}}$:

$$\zeta \mapsto e^{i\frac{2\pi l}{d}}\zeta, \quad l = 0, \ldots, d-1.$$

These rotations can be viewed as linear fractional transformations given by (1.2.8)-(1.2.9) with $z = e^{i\frac{\pi l}{d}}$, $w = 0$, $l = 0, \ldots, d-1$. Geometrically, C_d corresponds to the group of rotations $R_{2\pi l/d, N}$, $l = 0, \ldots, d-1$, with vertical axis of rotation through the north and south poles. Since cyclic groups are generated by a single element, it is clear by Theorem 1.2.1 and (1.2.7) that any finite cyclic subgroup of order d in $\mathcal{M}_0(\hat{\mathbf{C}})$ is conjugate to C_d.

If we adjoin to C_d the linear fractional transformation $\zeta \mapsto 1/\zeta$, with $z = 0$, $w = i$ in (1.2.8)-(1.2.9), corresponding to the half-turn R_{π, e_0}, $e_0 = (1, 0, 0)$, we obtain the *dihedral Möbius group* D_d of order $2d$:

$$\zeta \mapsto e^{i\frac{2\pi l}{d}}\zeta, \frac{e^{-i\frac{2\pi l}{d}}}{\zeta}, \quad l = 0, \ldots, d-1. \tag{1.2.11}$$

In the previous section, D_d was realized as the symmetry group of a d-prism. In view of Theorem 1.2.1, it is more appropriate to realize D_d as the symmetry group of Klein's *dihedron*, the regular spherical polyhedron with two hemispherical faces, d spherical edges, and d vertices distributed equidistantly along the equator of S^2. If we fix one vertex at e_0 then the half-turn R_{π, e_0} above is a symmetry of the dihedron that interchanges the two faces. We also see that in (1.2.11) the first group of linear fractional transformations corresponds to $z = e^{i\frac{\pi l}{d}}$, $w = 0$, $l = 0, \ldots, d-1$, and the second corresponds to $z = 0$, $w = ie^{i\frac{\pi l}{d}}$, $l = 0, \ldots, d-1$.

As a straightforward generalization, we inscribe a Platonic solid P in S^2, apply Theorem 1.2.1, and obtain a finite Möbius group G isomorphic with the symmetry group of P. Since dual pairs of Platonic solids have the same symmetry group, we may restrict ourselves to the tetrahedron, octahedron, and the icosahedron. We call the corresponding groups *tetrahedral*, *octahedral*, and *icosahedral Möbius groups*. The dihedral Möbius group discussed above can be considered as a member of this family of groups if we replace P with its spherical tessellation obtained by projecting P radially from the origin to S^2. In what follows, we call these configurations *spherical Platonic tesselations*. By Theorem 1.2.1 and (1.2.7), the Möbius groups obtained in two different ways of inscribing the same Platonic solid into S^2 are con-

jugate subgroups in $\mathcal{M}_0(\hat{\mathbf{C}})$. Thus we can choose our spherical Platonic tesselations in convenient positions in S^2.

We choose our regular tetrahedron such that its vertices are alternate vertices of the cube, and the cube is inscribed in S^2 such that its faces are parallel to the coordinate planes. We also agree that the first octant contains a vertex of the tetrahedron. This vertex must be

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right). \tag{1.2.12}$$

The edges of the tetrahedron are diagonals of the faces of the cube, one diagonal for each face. Hence the three coordinate axes go through the midpoint of the three opposite pairs of edges of the tetrahedron. The three half-turns around these axes are symmetries of the tetrahedron. Applying these half-turns to (1.2.12), we obtain the three remaining vertices

$$\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right). \tag{1.2.13}$$

As noted above, the half-turn R_{π,e_0}, $e_0 = (1,0,0)$, corresponds to $\zeta \mapsto 1/\zeta$. In a similar vein, the half-turn R_{π,e_1}, $e_1 = (0,1,0)$, corresponds to $\zeta \mapsto -1/\zeta$. Finally, $R_{\pi,N}$ corresponds to $\zeta \mapsto -\zeta$ (with $z = i$ and $w = 0$ in (1.2.8)-(1.2.9)). Adjoining the identity, we have

$$\zeta \mapsto \pm\zeta, \pm\frac{1}{\zeta}.$$

This gives the dihedral Möbius group D_2 of order 4. (Compare this with Problem 1.18.) We see that D_2 is a subgroup of the tetrahedral Möbius group T.

The four lines through the origin and (1.2.12)-(1.2.13) connecting a vertex and the centroid of the opposite face of the tetrahedron are axes of symmetry rotations with angles $2\pi/3$ and $4\pi/3$. Substituting the vectors in (1.2.12)-(1.2.13) as (x_0, x_1, x_2) in (1.2.9) and setting $\theta = 2\pi/3, 4\pi/3$, we obtain $z = (\pm 1 + i)/2, w = \pm(1+i)/2$, and $z = (\pm 1 - i)/2, w = \pm(1-i)/2$. Substituting these into (1.2.8) we arrive at the following 8 linear fractional transformations

$$\zeta \mapsto \pm i\frac{\zeta+1}{\zeta-1}, \pm i\frac{\zeta-1}{\zeta+1}, \pm\frac{\zeta+i}{\zeta-i}, \pm\frac{\zeta-i}{\zeta+i}.$$

Putting everything together, we obtain the 12 elements of the tetrahedral Möbius group T:

$$\zeta \mapsto \pm\zeta, \pm\frac{1}{\zeta}, \pm i\frac{\zeta+1}{\zeta-1}, \pm i\frac{\zeta-1}{\zeta+1}, \pm\frac{\zeta+i}{\zeta-i}, \pm\frac{\zeta-i}{\zeta+i}. \tag{1.2.14}$$

We position the octahedron in S^2 such that its vertices are the six intersections of the coordinate axes with S^2. This octahedron is homothetic to the intersection of the tetrahedron above and its dual. As noted in the previous section, the octahedral group is generated by the tetrahedral group

and a symmetry of the octahedron that interchanges the tetrahedron with its dual. An example for the latter is the quarter-turn around the vertical axis. This quarter-turn corresponds to the linear fractional transformation $\zeta \mapsto i\zeta$ characterized by $z = e^{i\frac{\pi}{4}}$ and $w = 0$. Thus the 24 elements of the octahedral Möbius group O are:

$$\zeta \mapsto i^l\zeta, \frac{i^l}{\zeta}, i^l\frac{\zeta+1}{\zeta-1}, i^l\frac{\zeta-1}{\zeta+1}, i^l\frac{\zeta+i}{\zeta-i}, i^l\frac{\zeta-i}{\zeta+i}, \quad l = 0, 1, 2, 3.$$

Finally, we work out the icosahedral Möbius group I. We inscribe the icosahedron in S^2 such that the north and south poles become vertices. The icosahedron is made up of a northern and southern pentagonal pyramid separated by a pentagonal antiprism. The rotations S^j, $j = 0, \ldots, 4$, $S = R_{2\pi/5,N}$, are symmetries of the icosahedron and they correspond to the linear fractional transformations

$$S^j : \ \zeta \mapsto \omega^j\zeta, \quad j = 0, \ldots, 4, \tag{1.2.15}$$

where $\omega = e^{i\frac{2\pi}{5}}$ is a fifth root of unity. We still have the freedom to rotate the icosahedron around the vertical axis for a convenient position. We fine tune the position of the icosahedron by agreeing that the *second* coordinate axis must go through the midpoint of one of the cross edges of the pentagonal antiprism. The half-turn U around this axis thus becomes a symmetry of the icosahedron. As noted above, this half-turn corresponds to the linear fractional transformation

$$U : \ \zeta \mapsto -\frac{1}{\zeta}.$$

The rotations S and U do not generate the entire symmetry group of the icosahedron because they both leave the equator invariant. For another generator we choose the half-turn V whose axis of rotation is perpendicular to the axis of rotation of U and passes through the midpoint of an edge in the base of the upper pentagonal pyramid (Figure 18). U and V are commuting half-turns since their axes of rotation are perpendicular (Problem 1.5 (d)). The composition $W = UV$ is also a half-turn ($W^2 = (UV)^2 = UVUV = U^2V^2 = I$), and the axis of rotation is perpendicular to those of U and V. With the identity I, U, V, and W form a dihedral subgroup D_2 of the icosahedral group I. To obtain the linear fractional transformations corresponding to V and W, we need to determine the axis of V.

Lemma 1.2.3. *The vertices of the icosahedron stereographically projected to* $\hat{\mathbf{C}}$ *are*

$$0, \infty, \omega^j(\omega + \omega^4), \omega^j(\omega^2 + \omega^3), \ j = 0, \ldots, 4, \tag{1.2.16}$$

where $\omega = e^{i\frac{2\pi}{5}}$.

PROOF. The poles correspond to 0 and ∞. The remaining ten vertices of the pentagonal antiprism projected to $\hat{\mathbf{C}}$ appear in two groups of five points

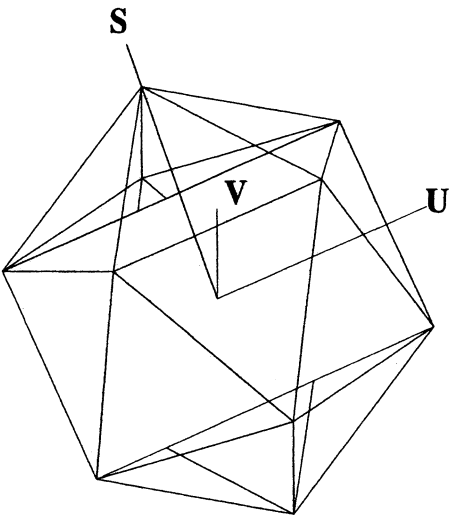


Figure 18.

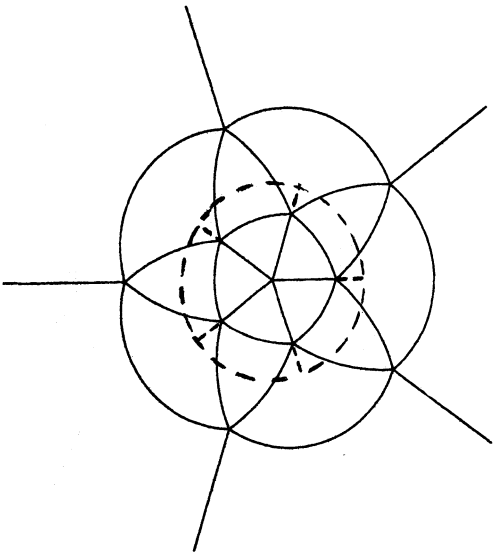


Figure 19.

equidistantly and *alternately* distributed in two concentric circles (Figure 19). As in Section 1.1, we think of the Pacioli's model of the icosahedron as being the convex hull of three mutually perpendicular golden rectangles. Assume now that we have a golden rectangle that contains the north pole as a vertex. The two sides of this golden rectangle emanating from the north pole can be extended to $\hat{\mathbf{C}}$, and give one projected vertex on each of

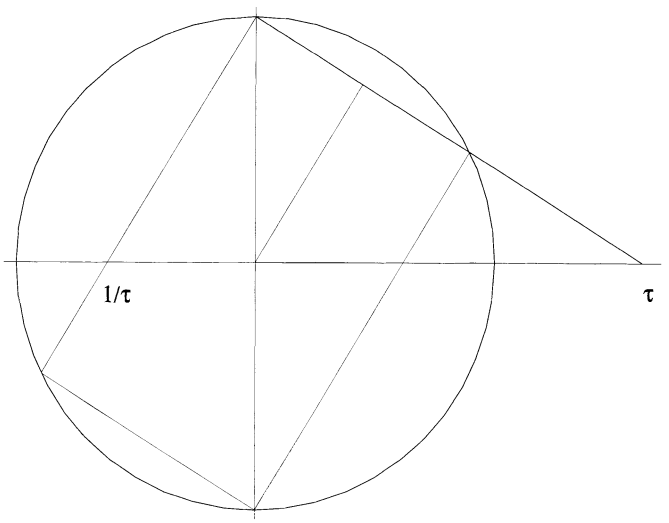


Figure 20.

the concentric circles. Considering similar triangles on the plane spanned by this golden rectangle, we see that the radii of the two concentric circles are the golden section τ and its reciprocal $1/\tau$ (Figure 20).

To rewrite τ and $1/\tau$ in terms of ω, we first observe that the regular pentagon P_5 inscribed in the unit circle $S^1 \subset \mathbf{C}$ has vertices ω^j, $j = 0, \ldots, 4$. By definition, τ is the ratio of a diagonal and a side length of P_5. Choosing the vertical diagonal and the vertical side, we have

$$\tau = \frac{|\omega - \omega^4|}{|\omega^2 - \omega^3|} = \frac{\omega - \omega^4}{\omega^2 - \omega^3},$$

where the last equality is due to both complex numbers $\omega - \omega^4$ and $\omega^2 - \omega^3$ being purely imaginary with positive imaginary parts. Simple arithmetic using $\omega^5 = 1$ now gives

$$\tau = -(\omega^2 + \omega^3) \text{ and } \frac{1}{\tau} = \omega + \omega^4. \tag{1.2.17}$$

Finally, the 5-fold symmetry of the projected vertices is given by multiplication by ω^j, $j = 0, \ldots, 4$. (1.2.16) follows (Fig. 19).

By the definition of the half-turn V and by Lemma 1.2.2, the axis of rotation of V passes through the midpoint of the segment connecting $h^{-1}(\omega^2(\omega^2+\omega^3))$ and $h^{-1}(\omega^3(\omega^2+\omega^3))$. To see this, using the 3-dimensional Figure 18, locate in Figure 19 (the projection of) the midpoint of the edge the axis of rotation of U passes through (on the second axis), and relate this point to $\omega^2(\omega^2 + \omega^3)$ and $\omega^3(\omega^2 + \omega^3)$.

By (1.2.6), this midpoint is

$$\frac{1}{2}h^{-1}(\omega^2(\omega^2 + \omega^3)) + \frac{1}{2}h^{-1}(\omega^3(\omega^2 + \omega^3))$$

$$= \left(\frac{\tau^2}{\tau^2+1}, 0, \frac{\tau^2-1}{\tau^2+1} \right),$$

where we used (1.2.17) repeatedly. Since $\tau^2 - 1 = \tau$, normalizing, we obtain that $V = R_{\pi,x}$, where

$$x = \left(\frac{\tau}{\sqrt{\tau^2+1}}, 0, \frac{1}{\sqrt{\tau^2+1}} \right). \tag{1.2.18}$$

Using the notations in (1.2.9) of Theorem 1.2.1 (with $\theta = \pi$), we thus have

$$z = \frac{i}{\sqrt{\tau^2+1}} \text{ and } w = \frac{i\tau}{\sqrt{\tau^2+1}}. \tag{1.2.19}$$

The common denominator can be rewritten as

$$\sqrt{\tau^2+1} = -i(\omega - \omega^4).$$

This is because

$$(\omega - \omega^4)^2 = \omega^2 + \omega^3 - 2 = -(\tau + 2) = -(\tau^2 + 1),$$

where we used (1.2.17) and the defining equality for τ. Using this in (1.2.19), we obtain

$$z = -\frac{1}{\omega - \omega^4} = \frac{\omega^2 - \omega^3}{\sqrt{5}} \text{ and } w = \frac{\omega^2 + \omega^3}{\omega - \omega^4} = \frac{\omega - \omega^4}{\sqrt{5}}. \tag{1.2.20}$$

Here, once again, we used (1.2.17) repeatedly along with $2\tau = 1 + \sqrt{5}$.

By (1.2.8)-(1.2.9), the linear fractional transformation corresponding to V is finally written as

$$V: \ \zeta \mapsto \frac{(\omega^2 - \omega^3)\zeta + (\omega - \omega^4)}{(\omega - \omega^4)\zeta - (\omega^2 - \omega^3)}. \tag{1.2.21}$$

Remark. To describe the icosahedral Möbius group explicitly, in his Icosahedron Book, Klein [1] makes an argument that involves the angle of the axis of the rotation of V with a coordinate axis. In the text above we adopted Schläfli's philosophy and expressed all metric properties in terms of the golden section τ and ω.

Composing V with $U : \zeta \mapsto -1/\zeta$, we obtain the linear fractional transformation corresponding to the half-turn $W = UV$:

$$W: \ \zeta \mapsto \frac{-(\omega - \omega^4)\zeta + (\omega^2 - \omega^3)}{(\omega^2 - \omega^3)\zeta + (\omega - \omega^4)}. \tag{1.2.22}$$

Making all possible combinations of these linear fractional transformations with (1.2.15) we finally arrive at the 60 elements of the icosahedral Möbius group I:

$$\zeta \mapsto \omega^j \zeta, \ -\frac{1}{\omega^j \zeta},$$

$$\omega^j \frac{-(\omega-\omega^4)\omega^l\zeta+(\omega^2-\omega^3)}{(\omega^2-\omega^3)\omega^l\zeta+(\omega-\omega^4)}, \tag{1.2.23}$$
$$\omega^j \frac{(\omega^2-\omega^3)\omega^l\zeta+(\omega-\omega^4)}{(\omega-\omega^4)\omega^l\zeta-(\omega^2-\omega^3)}, \quad j,l=0,\dots,4.$$

This completes our list of finite Möbius groups.

Recall that, given a finite Möbius group $G \subset \mathcal{M}_0(\hat{\mathbf{C}})$, the inverse image G^* of G under the homomorphism $SU(2) \to \mathcal{M}_0(\hat{\mathbf{C}})$ is called the *binary group associated to* G. Taking inverse images, from the finite Möbius groups just constructed, we obtain the *binary dihedral group* D_d^*, the *binary tetrahedral group* T^*, the *binary octahedral group* O^*, and the *binary icosahedral group* I^*. For cyclic groups, we have $C_d^* \cong C_{2d}$. We conclude this section by showing that, up to conjugation, these are all the finite subgroups of $SU(2)$. (The case of cyclic subgroups in $SU(2)$ is somewhat special; we have to allow a cyclic subgroup to have odd order.) A convenient way to describe these groups is by using *quaternions*. (For a concise summary, see Berger [1], Section 8.9.)

Let $\mathbf{H}$ denote the skew-field of quaternions with canonical basis $\{1,i,j,k\}$. A typical element in $\mathbf{H}$ is $a+bi+cj+dk$, $a,b,c,d \in \mathbf{R}$, and quaternionic multiplication obeys the relations

$$i^2=j^2=k^2=-1, \quad ij=-ji=k, \quad jk=-kj=i, \quad ki=-ik=j,$$

and it is extended linearly to $\mathbf{H}=\mathbf{R}^4$.

It is easy to check that the group of unit quaternions $S^3 \subset \mathbf{H}$ is isomorphic with $SU(2)$ with the isomorphism that associates to the quaternion $z+jw \in \mathbf{H}$, $|z|^2+|w|^2=1$, $z,w \in \mathbf{C}$, the matrix (1.2.4). (Notice that, setting $z=a+bi$ and $w=c+di$, $a,b,c,d \in \mathbf{R}$, we have $z+jw=a+bi+cj-dk$.) In what follows, we *identify* S^3 and $SU(2)$ with this isomorphism. In each case of a finite Möbius group $G \subset \mathcal{M}_0(\hat{\mathbf{C}})$ in our list, we can put together the elements of the binary group G^* associated to G. G^* consists of quaternions $\pm(z+jw)$, where z,w correspond to the elements of G via (1.2.8)-(1.2.9). We obtain the following list of finite subgroups of S^3:

1. Cyclic group of order d:

$$C_d=\{e^{i\frac{2\pi l}{d}} \mid l=0,\dots,d-1\}. \tag{1.2.24}$$

2. Binary dihedral group of order $4d$:

$$D_d^*=\{e^{i\frac{\pi l}{d}}, l=0,\dots,2d-1\} \cup j\cdot\{e^{i\frac{\pi l}{d}}, l=0,\dots,2d-1\}. \tag{1.2.25}$$

3. Binary tetrahedral group:

$$T^*=D_2^* \cup \left\{\frac{\pm 1 \pm i \pm j \pm k}{2}\right\}. \tag{1.2.26}$$

(The nontrivial elements of the binary quadratic subgroup D_2^* correspond to the half-turns around the coordinate axes.)

4. Binary octahedral group:

$$O^* = T^* \cup e^{i\pi/4} \cdot T^*. \tag{1.2.27}$$

(The quarter-turn around the first axis corresponds to the quaternions $\pm e^{i\frac{\pi}{4}}$.)

5. Binary icosahedral group:

$$\begin{aligned} I^* = \{&\pm\omega^l, \pm j\omega^l, \\ &\pm\frac{1}{\sqrt{5}}\Big(-\omega^{3l}(\omega-\omega^4)+j\omega^{2l}(\omega^2-\omega^3)\Big)\omega^{3k}, \\ &\pm\frac{1}{\sqrt{5}}\Big(\omega^{3l}(\omega^2-\omega^3)+j\omega^{2l}(\omega-\omega^4)\Big)\omega^{3k}, \\ &k,l=0,\ldots,4\} \end{aligned} \tag{1.2.28}$$

(The quaternion ω^l corresponds to the diagonal matrix in $SU(2)$ with diagonal entries $\omega^{l/2}$ and $\omega^{-l/2}$. Moreover, $\omega^{l/2} = (-1)^l\omega^{3l}$ and $\omega^{-l/2} = (-1)^l\omega^{2l}$.)

These groups can be visualized by the *Clifford decomposition* of S^3. First, S^3 can be parametrized by the quaternion $z + jw$, where the complex parameters $z, w \in \mathbf{C}$ satisfy $|z|^2 + |w|^2 = 1$. Hence, we can introduce "isoparametric coordinates" $z = \cos(t)e^{i\theta}$ and $w = \sin(t)e^{i\phi}$, $0 \le t \le \pi/2$, $0 \le \theta, \phi < 2\pi$. For fixed t, we consider the level set

$$T_t = \{(z,w) \in S^3 \subset \mathbf{C}^2 \,|\, |z|^2 - |w|^2 = \cos(2t)\}.$$

For $t = 0, \pi/2$, T_0 and $T_{\pi/2}$ are circles cut out from S^3 by the coordinate planes spanned by the first two and the last two axes in $\mathbf{R}^4 = \mathbf{C}^2$. For $0 < t < \pi/2$, T_t is a torus. We call T_t the Clifford torus with parameter t. We will think of T_t as the square $[0, 2\pi] \times [0, 2\pi]$ with the opposite sides identified. θ and ϕ are natural parameters on this square. We now describe how the cyclic and binary groups above fit in the Clifford decomposition of $S^3 = SU(2)$.

The cyclic group C_d given in (1.2.24) lies entirely in T_0, and its elements are the vertices of a regular d-sided polygon inscribed in T_0. The binary dihedral group D_d^* given in (1.2.25) consists of two copies (of the vertices) of the regular $2d$-sided polygon, one inscribed in T_0, and the other in $T_{\pi/2}$. Figure 21 shows D_4^*. The binary tetrahedral group T^* has D_2^* as a subgroup and the remaining 16 elements in (1.2.26) lie in the middle Clifford torus $T_{\pi/4}$. This is because

$$z = \frac{\pm 1 \pm i}{2} \quad \text{and} \quad w = \frac{\pm 1 \mp i}{2}$$

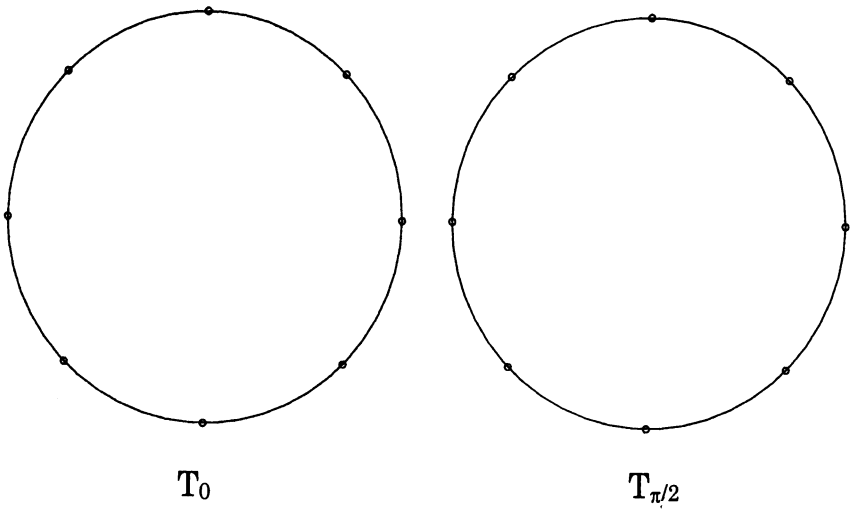


Figure 21.

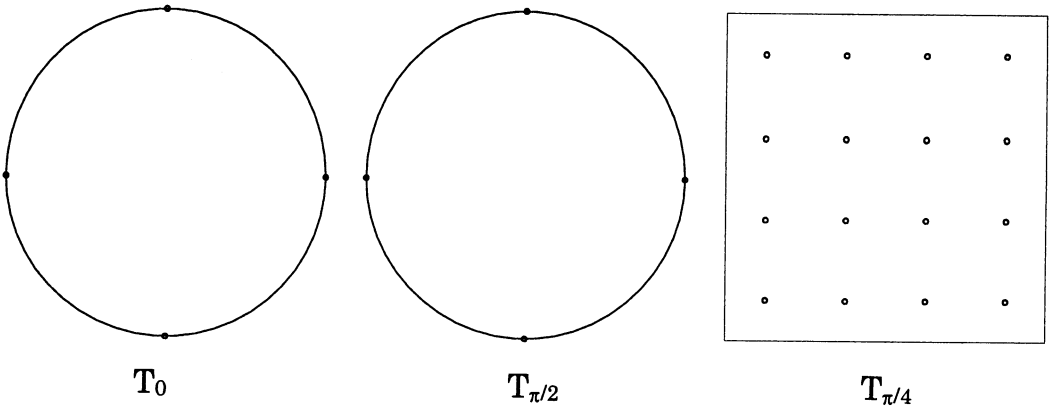


Figure 22.

so that

$$|z|^2 - |w|^2 = \left|\frac{\pm 1 \pm i}{2}\right|^2 - \left|\frac{\pm 1 \mp i}{2}\right|^2 = \frac{1}{2} - \frac{1}{2} = 0.$$

Working out the arguments of $z = e^{i\theta}/\sqrt{2}$ and $w = e^{i\phi}/\sqrt{2}$, we see that the parameter values of the corresponding points are $\theta = (2k+1)\pi/4$, $\phi = (2l+1)\pi/4$, $k, l = 0, \ldots, 3$ (Figure 22). The binary octahedral group O^* given in (1.2.27) contains T^* as an index two subgroup. Multiplication by $e^{i\pi/4}$ has the effect of adding $\pi/4$ to the parameters θ and ϕ obtained

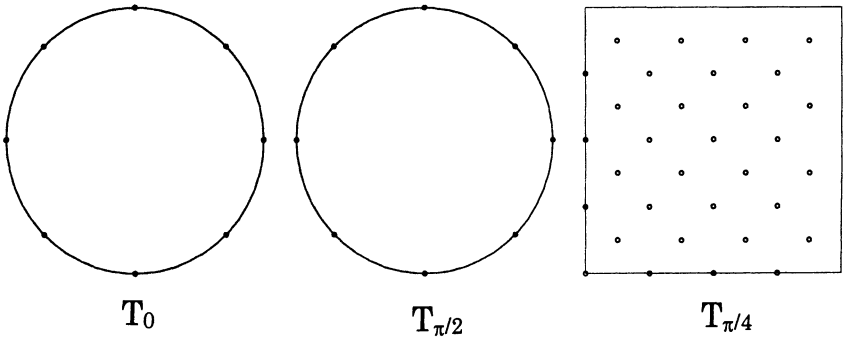


Figure 23.

for the points corresponding to the elements of T^*. Note that D_4^* becomes a subgroup of O^* (Figure 23). The binary icosahedral group I^* in (1.2.28) needs a bit of computation. It is clear that the elements $\pm\omega^l$ and $\pm j\omega^l$, $l = 0, \ldots, 4$, make up the vertices of two copies of a regular decagon, one inscribed in T_0, the other in $T_{\pi/2}$. These account for 20 elements of I^*. For the remaining 100 elements, we have

$$\begin{aligned} |z|^2 - |w|^2 &= \pm\frac{1}{5}(|\omega - \omega^4|^2 - |\omega^2 - \omega^3|^2) \\ &= \mp\frac{1}{5}((\omega - \omega^4)^2 - (\omega^2 - \omega^3)^2) \\ &= \pm\frac{1}{5}(\omega + \omega^4 - \omega^2 - \omega^3) \\ &= \pm\frac{1}{5}\left(\frac{1}{\tau} + \tau\right) = \frac{2\tau - 1}{5} = \pm\frac{1}{\sqrt{5}}. \end{aligned}$$

In this computation we used the fact that $\omega - \omega^4$ and $\omega^2 - \omega^3$ are both purely imaginary, the equalities in (1.2.17), and $\tau = (1 + \sqrt{5})/2$.

We see that these elements (in two groups of 50) are on the two Clifford tori $T_{t_\pm}$, where $\cos(2t_\pm) = \pm 1/\sqrt{5}$. Calculating the arguments for T_{t_+}, we obtain

$$\left(\frac{3\pi}{2} + \frac{2k\pi}{5}, \frac{\pi}{2} + \frac{2l\pi}{5}\right), \quad \left(\frac{\pi}{2} + \frac{2k\pi}{5}, \frac{3\pi}{2} + \frac{2l\pi}{5}\right),$$

where k, l are integers modulo 5. Similarly, on T_{t_-}, we have

$$\left(\frac{\pi}{2} + \frac{2k\pi}{5}, \frac{\pi}{2} + \frac{2l\pi}{5}\right), \quad \left(\frac{3\pi}{2} + \frac{2k\pi}{5}, \frac{3\pi}{2} + \frac{2l\pi}{5}\right),$$

where again k, l are integers modulo 5. The entire binary icosahedral group I^* is depicted in Figure 24. A 3-dimensional view of the binary icosahedral group is in the illustration on page 37.

Remark. To relate the binary icosahedral group I^* to the binary tetrahedral group T^*, it is sometimes more convenient to realize the icosahedron in

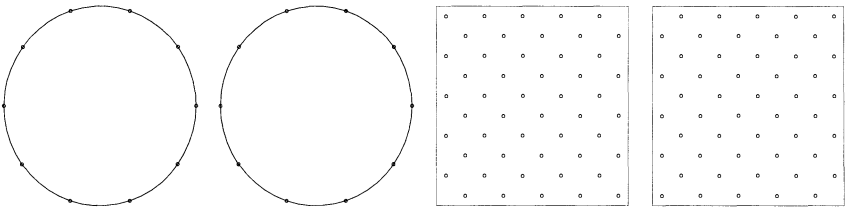


Figure 24.

another position. Recall from Section 1.1 the compound model of the five circumscribed tetrahedra whose faces enclose the icosahedron. As noted there, the five tetrahedra constitute an orbit of an order 5 symmetry rotation of the icosahedron. A rotation generating the orbit has axis in the coordinate plane perpendicular to the first axis and the slope of the axis is the golden section τ. A simple computation in the use of (1.2.8)-(1.2.9) shows that the pair of quaternions that corresponds to this rotation is

$$\pm\frac{1}{2}\left(\tau + i + \frac{j}{\tau}\right). \tag{1.2.29}$$

The symmetry groups of the five tetrahedra are clearly subgroups of the symmetry group of the icosahedron, and the symmetries of the icosahedron permute the five tetrahedra. As a byproduct, we see again that the symmetry group of the icosahedron is isomorphic with the alternating group $\mathcal{A}_5$ on five letters. Retaining the earlier notation for this conjugate subgroup, we obtain

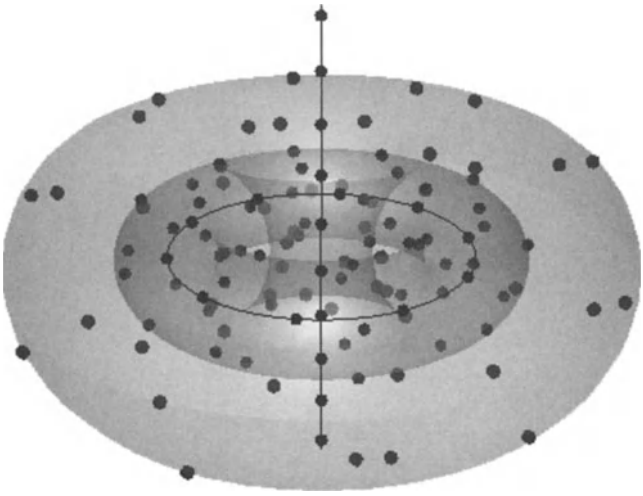
$$I^* = T^* \cup q \cdot T^* \cup q^2 \cdot T^* \cup q^3 \cdot T^* \cup q^4 \cdot T^*, \tag{1.2.30}$$

where q is given in (1.2.29).

We are now ready to state the main result of this section.

Theorem 1.2.4. *Any finite subgroup of $S^3 = SU(2)$ is either cyclic, or conjugate to one of the binary subgroups D_d^*, T^*, O^*, or I^*.*

PROOF. Let $G \subset S^3 = SU(2)$ be a finite subgroup with corresponding subgroup G_0 in $SU(2)/\{\pm I\} = \mathcal{M}_0(\hat{\mathbf{C}})$. Let $G^* \subset S^3$ be the inverse image of G_0 under the canonical projection $SU(2) \to SU(2)/\{\pm I\}$. Clearly, $G \subset G^*$. By Theorem 1.2.1, G_0 is isomorphic with a finite subgroup of $SO(3)$. If $G = G^*$ then G^* is the double cover of the group G_0. In this case the statement follows from Theorem 1.1.2. Thus, we need only study the case when $G \neq G^*$. In this case, G is of index 2 in G^* and G and G_0 are isomorphic. We first claim that G is of odd order. Assume, on the contrary, that G has even order. It is well-known in group theory that G must contain an element of order 2. Since the only order 2 element in $SU(2)$ is $-I$, it must be contained in G. But $\{\pm I\}$ is the kernel of the canonical projection and this is in contradiction to $G \neq G^*$. Thus G has odd order. G is isomorphic



with G_0 and, as noted above, the latter has an isomorphic copy in $SO(3)$. The odd order subgroups in $SO(3)$ are cyclic as follows again from Theorem 1.1.2.

By definition, quotients of S^3 by finite subgroups give all 3-dimensional homogeneous spherical space forms. S^3/C_d, $d \geq 2$, is the so-called *lens space* $L(d;1)$. (The lens spaces $L(p,q)$ are the most general spherical space forms. For a geometric account, see Thurston [1], pp. 37-38.) Based on the solids whose symmetry groups they are derived from, S^3/D_d^*, $d \geq 1$, are called *prism manifolds*, S^3/T^* is the *tetrahedral manifold*, S^3/O^* is the *octahedral manifold*, and S^3/I^* is the *icosahedral manifold*. (For a general reference on polyhedral manifolds, see Scott [1].)

Corollary 1.2.5. *The 3-dimensional homogeneous spherical space forms are the lens spaces $L(d;1)$, $d \geq 2$; the prism manifolds S^3/D_d^*, $d \geq 1$; and the tetrahedral, octahedral, and icosahedral manifolds.*

Remark. If K/H is a homogeneous space then H acts on the tangent space $T_o(K/H)$ at $o = \{H\}$ by the *isotropy representation*. (A standard reference on homogeneous spaces is Helgason [1].) Wolf [1] classified all homogeneous spaces K/H, where the action of the identity component H_0 of H on $T_o(K/H)$ is irreducible (no invariant linear subspace). Wang and Ziller [2] classified those homogeneous spaces K/H in which the action of H on the tangent space $T_o(K/H)$ is irreducible, but the action of H_0 is not. The simplest examples are $SO(3)/G$, where G are the symmetry groups of the Platonic solids ($K = SO(3)$ and $H = G$ with H_0 trivial).

For a beautiful geometric classification of all spherical 3-manifolds (homogeneous or not), see Thurston [1], Section 4.4.

1.3 Invariant Forms

The purpose of this preparatory section is twofold. First, we derive the polynomial invariants of the cyclic, dihedral, and binary polyhedral groups. These invariants will be used in Section 1.5 to construct isometric minimal imbeddings of 3-dimensional space forms into spheres. In addition, they will be used to define the polyhedral equations, crucial in Klein's theory of the icosahedron. Klein's theory is the subject of the Additional Topic at the end of this chapter. Second, we will prove that, up to conjugacy, the only finite Möbius groups are cyclic, dihedral, and the 3 polyhedral Möbius groups discussed in the previous section. We will use some elementary facts in projective geometry in what follows. A reference is Berger [1], Section 4.2. The proof of Theorem 1.3.1 uses the Riemann-Hurwitz relation. We put this (along with some supplementary material) in Problem 1.27. Farkas-Kra [1] is a standard reference for this topic.

The geometry of the spherical Platonic tessellations (including the dihedron) can be conveniently described by the so-called *characteristic triangle.* (For spherical tessellations we will use the spherical analogues of the Euclidean polyhedral concepts such as vertex, edge, face, etc. with obvious meanings.) Given a spherical Platonic tessellation, each *spherical flag* consisting of a vertex v, an edge e and a face f with $v \in e \subset f (\subset S^2)$ contains one characteristic triangle with vertices v_0, v_1, v_2, where $v_0 = v$, v_1 is the midpoint of the edge e, and v_2 is the centroid of the face f. We write the spherical angles of a characteristic triangle at the respective vertices as π/ν_0, π/ν_1, and π/ν_2, where ν_0, ν_1, ν_2 are integers ≥ 2. In fact, ν_0 is the number of faces meeting at a vertex, $\nu_1 = 2$ since v_1 is the midpoint of an edge, and ν_2 is the number of sides of a face in the tessellation. If the spherical tessellation is coming from a Platonic solid by radial projection then $\{\nu_2, \nu_0\}$ is the Schläfli-symbol of the solid. In the previous section we saw that a spherical Platonic tessellation is dihedral, or it is congruent to the tetrahedral, octahedral or icosahedral tessellation. For the dihedron, we have $\nu_0 = 2$, $\nu_2 = d$; for the tetrahedron, $\nu_0 = \nu_2 = 3$; for the octahedron, $\nu_0 = 4$, $\nu_2 = 3$; and, finally, for the icosahedron, $\nu_0 = 5$, $\nu_2 = 3$. It is easy to see (Problem 1.5) that reflections to the (linear span of the) sides of a characteristic triangle generate the extended symmetry group.

Each axis of a symmetry rotation of a spherical Platonic tessellation goes through a vertex, or the midpoint of an edge, or the centroid of a face. It follows that ν_0, ν_1, and ν_2 are the orders of the symmetry rotations with axes through the respective points. For example, a symmetry rotation around the midpoint of an edge is always a half-turn.

We look at this now from the point of view of the orbit structure of the symmetry group G of the tessellation. By the counting formula mentioned in Section 1.1, the number of points on the orbit $G(x)$ passing through a point $x \in S^2$ is $|G(x)| = |G|/|G_x|$, where G_x is the (cyclic) isotropy group at x. If the isotropy group is nontrivial then it is generated by a

(nontrivial) rotation, and, in this case, x must be a vertex, the midpoint of an edge, or the centroid of a face of the tessellation. On the other hand, by regularity, the set of all vertices form a single orbit, and the same is true for the set of midpoints of all the edges, and the set of centroids of all the faces. We conclude that we have 3 *exceptional orbits* of the action of G on S^2, where exceptional means that the number of elements on the orbit is less than the order of G. Since a symmetry rotation around a vertex has order ν_0, the exceptional orbit comprised by all the vertices has $V = |G|/\nu_0$ elements. In a similar vein, the orbit of the midpoints (of the edges) consists of $E = |G|/\nu_1$ elements, and the orbit of the centroids (of the faces) consists of $F = |G|/\nu_2$ elements.

All the other orbits are *principal*, i.e. the number of elements on the orbit is equal to $|G|$. By Euler's theorem for convex polyhedra (1.1.1), we have

$$V - E + F = \frac{|G|}{\nu_0} - \frac{|G|}{\nu_1} + \frac{|G|}{\nu_2} = 2. \tag{1.3.1}$$

Recall from the previous section that $SL(2, \mathbf{C})$ acts on $\mathbf{C}^2$ by matrix multiplication. For $g \in SL(2, \mathbf{C})$ in the matrix form given in (1.2.3), the action of g is given by $(z, w) \mapsto g \cdot (z, w) = (az + bw, cz + dw)$, where $(z, w) \in \mathbf{C}^2$ is considered as a column vector. Since the matrices in $SL(2, \mathbf{C})$ are nonsingular, this action extends to $\mathbf{C}P^1$ by setting $[z : w] \mapsto g \cdot [z : w] = [(az + bw) : (cz + dw)]$, where we used homogeneous coordinates on $\mathbf{C}P^1$. On the other hand, $\mathbf{C}P^1$ can be identified with the extended complex plane $\hat{\mathbf{C}}$ via the identification $[z : w] \mapsto \zeta = z/w$, $[z : w] \in \mathbf{C}P^1$. With this identification, the action of $SL(2, \mathbf{C})$ on $\mathbf{C}P^1$ is carried over to an action on $\hat{\mathbf{C}}$. In fact, $g \in SL(2, \mathbf{C})$ given in (1.2.3) acts on $\hat{\mathbf{C}}$ via the linear fractional transformation (1.2.1). This is because we have

$$\frac{az + bw}{cz + dw} = \frac{az/w + b}{cz/w + d} = \frac{a\zeta + b}{c\zeta + d}.$$

Restricting ourselves to finite groups, we conclude that the action of a finite Möbius group G on $\hat{\mathbf{C}}$ by linear fractional transformations corresponds to the ordinary action of the binary group $G^* \subset SL(2, \mathbf{C})$ on $\mathbf{C}^2$ by matrix multiplication.

A homogeneous polynomial $\xi : \mathbf{C}^2 \to \mathbf{C}$ is called a *form*. If ξ has degree p then homogeneity amounts to

$$\xi(\lambda z, \lambda w) = \lambda^p \xi(z, w), \quad \lambda, z, w \in \mathbf{C}.$$

Given any subgroup $G^* \subset SL(2, \mathbf{C})$, we say that ξ is G^*-*invariant* if there exists a *character* $\chi_\xi : G^* \to \mathbf{C} - \{0\}$, a homomorphism of G^* into the multiplicative group $\mathbf{C} - \{0\}$, such that

$$\xi \circ g = \chi_\xi(g) \cdot \xi, \quad g \in G^*.$$

Here $g \in SL(2,\mathbf{C})$ acts on $\mathbf{C}^2$ by matrix multiplication as above. χ_ξ is uniquely determined by ξ. We say that ξ is an *absolute invariant* of G^* if $\chi_\xi = 1$. In general, ξ is an absolute invariant of the subgroup $\ker \chi_\xi \subset G^*$. If G^* is finite (as in our study of the binary polyhedral groups) then χ_ξ maps into the unit circle $S^1 \subset \mathbf{C}-\{0\}$, and the quotient $G^*/\ker\chi_\xi \cong \operatorname{im}\chi_\xi \subset S^1$ is cyclic. This is because any finite subgroup of $\mathbf{C}-\{0\}$ is a cyclic subgroup of $S^1 \subset \mathbf{C}-\{0\}$ (Problem 1.24).

To each finite Möbius group $G \subset \mathcal{M}_0(\hat{\mathbf{C}})$ in our list, with corresponding binary Möbius group $G^* \subset SU(2)$, we will exhibit two forms ξ, η of degree $|G|$ with no common zeros such that ξ and η are both G^*-invariant and have the same character. For these ξ and η, we then define $q = q_G : \mathbf{C}P^1 \to \mathbf{C}P^1$ by

$$q([z:w]) = [\xi(z,w) : \eta(z,w)], \quad z,w \in \mathbf{C}.$$

Clearly q is well-defined and G^*-invariant. By the identification $\mathbf{C}P^1 = \hat{\mathbf{C}}$, it becomes a holomorphic map $q : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ with rational restriction to $\mathbf{C}$ (denoted by the same symbol). The latter is given by

$$q(\zeta) = \frac{\xi(z,w)}{\eta(z,w)} = \frac{\xi(\zeta,1)}{\eta(\zeta,1)}, \quad \zeta = z/w, \;\; z,w \in \mathbf{C}.$$

By the discussion on the group actions above, G^*-invariance of ξ and η (with common character $\chi_\xi = \chi_\eta$) implies that q is G-invariant. q is the G-invariant rational function we seek.

The cyclic group $G = C_d = \{\zeta \mapsto e^{i\frac{2\pi l}{d}} \mid l = 0,\dots,d-1\}$ does not fit in the general framework since it is not the symmetry group of a spherical Platonic tessellation. Although it is easy to obtain the general C_d-invariant rational function $q : \mathbf{C} \to \mathbf{C}$ by inspection, it is instructive to go through the details in this simple case. We first seek the most general C_d^*-invariant form ξ of degree d. $C_d^* = C_{2d}$ is cyclic so that the character χ_ξ is uniquely determined by its value on the generator $\epsilon = e^{i\frac{\pi}{d}}$. Since $\epsilon^{2d} = 1$, $\chi_\xi(\epsilon)$ must be a $2d$-th root of unity, i.e., $\chi_\xi(\epsilon) = \epsilon^m$, for some $m = 0,\dots,2d-1$. The condition of C_d^*-invariance for ξ reduces to

$$\xi(\epsilon z, \epsilon^{-1} w) = \epsilon^m \xi(z,w), \;\; z,w \in \mathbf{C}.$$

Being homogeneous of degree d, the typical monomial participating in ξ is $z^j w^{d-j}$, where $j = 0,\dots,d$. Substituting this into the equation above, we obtain $\epsilon^{2j} = \epsilon^{m+d}$. Since ϵ is a $2d$-th root of unity, we have

$$2d \,|\, m + d - 2j.$$

In particular, $d \,|\, m-2j$, or equivalently, $m - 2j = dl$ for some integer l. Hence $2d \,|\, d(l+1)$, and l must be odd. Inspecting the ranges of j and m, we see that $|m-2j| \le 2d$ so that $l = \pm 1$. We have $2j = m \pm d$ so that m and d have the same parity, $j = (m \pm d)/2$. For $l = 1$, we have $m \ge d$, and

for $l = -1$, $m \leq d$. The corresponding monomials are

$$z^{\frac{m \pm d}{2}} w^{d - \frac{m \pm d}{2}}.$$

Two linearly independent C_d^*-invariant forms of degree d having the same character (i.e., the same m) exist iff $m = d$, and, in this case, the general form is a linear combination of z^d and w^d. Since $\zeta = z/w$, the general C_d-invariant rational function q is a quotient of two linearly independent forms. We obtain that the most general C_d-invariant rational function is a linear fractional transformation applied to ζ^d. Notice that ζ^d vanishes on the fixed points 0 and ∞ of the rotations that make up C_d. Analytically, *the fixed points are the branch points of* q considered as a holomorphic d-fold branched covering $q : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$, $q(\zeta) = \zeta^d$, $\zeta \in \hat{\mathbf{C}}$.

This last remark gives us a clue as to how to obtain invariant forms for spherical Platonic tessellations. First, we discuss the dihedron with the special position from Section 1.2. Recall that the vertices of a dihedron are the d points distributed uniformly along the equator of S^2 with $e_0 = (1, 0, 0)$ being a vertex. On $\hat{\mathbf{C}}$, these points constitute the roots of the equation $\zeta^d = 1$. We see that the most general degree d form that vanishes on these vertices is a constant multiple of $z^d - w^d$. We set

$$\alpha(z, w) = \frac{z^d - w^d}{2}. \tag{1.3.2}$$

In a similar vein, the most general degree d form that vanishes on the midpoints of the edges of the dihedron is a constant multiple of $z^d + w^d$. We define

$$\beta(z, w) = \frac{z^d + w^d}{2}. \tag{1.3.3}$$

Finally, the two centroids of the hemispherical faces correspond to 0 and ∞, and we put

$$\gamma(z, w) = zw. \tag{1.3.4}$$

We say that the forms α, β, and γ *belong to the dihedron.* Each of these three forms are D_d^*-invariant with $\chi_\alpha = \chi_\beta$, and $\chi_\gamma = \pm 1$ with $+1$ corresponding to the cyclic kernel $\ker \chi_\gamma = C_d^* \subset D_d^*$. The forms α, β, and γ are *algebraically dependent*:

$$\alpha^2 - \beta^2 + \gamma^d = \left(\frac{z^d - w^d}{2}\right)^2 - \left(\frac{z^d + w^d}{2}\right)^2 + (zw)^d = 0. \tag{1.3.5}$$

Notice that the degrees of α, β, and γ are $|D_d|/\nu_0$, $|D_d|/\nu_1$, and $|D_d|/\nu_2$ respectively, and consequently, the exponents in (1.3.5) are ν_0, ν_1, and ν_2.

It is worthwhile to generalize some of these properties of the forms of the dihedron. In fact, in each of the remaining cases of Platonic tessellations we will have 3 forms ξ_0, ξ_1, and ξ_2 that will be said to belong to the tessellation in the sense that ξ_0 vanishes on the projected vertices, ξ_1 vanishes on the

projected midpoints of the edges, and ξ_2 vanishes on the projected centroids of the faces. ξ_0, ξ_1, and ξ_2 are invariants of the corresponding binary Möbius group G^*, and they have degrees $|G|/\nu_0$, $|G|/\nu_1$, and $|G|/\nu_2$.

Remark. The developments here are essentially contained in Klein [1]. In trying to make an up-to-date treatment of the subject, we adopted a number of changes while retaining as much classical terminology as possible. For example, our notation for the forms ξ_0, ξ_1, ξ_2 differs from Klein's; our ξ_j, $j = 0, 1, 2$, vanishes on the centroids of the j-dimensional cells of the Platonic solid.

We claim that, in general, the forms $\xi_0^{\nu_0}$, $\xi_1^{\nu_1}$, $\xi_2^{\nu_2}$ (all of degree $|G|$) are *linearly dependent.* To see this, recall that, away from the three exceptional orbits of G on S^2 (the zero sets of ξ_0, ξ_1, and ξ_2), all orbits are principal. Given a principal orbit, we can find a complex ratio $\mu_0 : \mu_1$ such that $\mu_0\xi_0^{\nu_0} + \mu_1\xi_1^{\nu_1}$ vanishes at one point of this orbit. By invariance, this linear combination vanishes at each point of the orbit. Similarly, for another ratio $\mu_1 : \mu_2$, $\mu_1\xi_1^{\nu_1} + \mu_2\xi_2^{\nu_2}$ vanishes on the same orbit. These two linear combinations are polynomials of degree $|G|$ and both vanish on a principal orbit containing $|G|$ points. It follows that they must be constant multiples of each other. We obtain that

$$\lambda_0\xi_0^{\nu_0} + \lambda_1\xi_1^{\nu_1} + \lambda_2\xi_2^{\nu_2} = 0 \tag{1.3.6}$$

for some $\lambda_0 : \lambda_1 : \lambda_2$. Notice that, for the dihedron, this relation reduces to (1.3.5).

We finally claim that *every G^*-invariant form ξ can be written as a polynomial in ξ_0, ξ_1, ξ_2.* We proceed by induction with respect to the degree of ξ, and exhibit a factor of ξ, a polynomial in ξ_0, ξ_1, ξ_2. Consider the zero set of ξ. By invariance of ξ, this set is G-invariant, the union of some G-orbits. If there is an exceptional orbit among these, then one of ξ_0, ξ_1, or ξ_2 divides ξ. Otherwise, there must be a principal orbit on which ξ vanishes. As above, for a complex ratio $\mu_0 : \mu_1$, the linear combination $\mu_0\xi_0^{\nu_0} + \mu_1\xi_1^{\nu_1}$ vanishes on this principal orbit, and therefore this combination must be a factor of ξ. The claim follows.

We now consider the tetrahedron. Applying the stereographic projection h to the vertices (1.2.12)-(1.2.13) of our tetrahedron, (1.2.5) immediately gives

$$h\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \pm\frac{1+i}{\sqrt{3}-1}, \quad h\left(\pm\frac{1}{\sqrt{3}}, \mp\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) = \pm\frac{1-i}{\sqrt{3}+1}.$$

A form Φ of degree 4 that vanishes on these points can be obtained by multiplying out the corresponding linear factors:

$$\Phi(z, w) = \left(z - \left(\frac{1+i}{\sqrt{3}-1}\right)w\right)\left(z + \left(\frac{1+i}{\sqrt{3}-1}\right)w\right)$$

$$\begin{aligned}
&\times \left(z - \left(\frac{1-i}{\sqrt{3}+1}\right)w\right)\left(z + \left(\frac{1-i}{\sqrt{3}+1}\right)w\right) \qquad (1.3.7)\\
&= \left(z^2 - \left(\frac{1+i}{\sqrt{3}-1}\right)^2 w^2\right)\left(z^2 - \left(\frac{1-i}{\sqrt{3}+1}\right)^2 w^2\right)\\
&= z^4 - 2\sqrt{3}iz^2w^2 + w^4.
\end{aligned}$$

The vertices of the dual tetrahedron are the antipodals of the vertices of the original tetrahedron. Since $h(-x) = -1/\overline{h(x)}$, $x \in S^2$, (Problem 1.25), these vertices projected to $\hat{\mathbf{C}}$ are

$$\mp\frac{1+i}{\sqrt{3}+1}, \ \mp\frac{1-i}{\sqrt{3}-1}.$$

The corresponding form Ψ of degree 4 is

$$\Psi = z^4 + 2\sqrt{3}iz^2w^2 + w^4. \qquad (1.3.8)$$

The vertices of the tetrahedron and its dual are the two alternate sets of vertices of the circumscribed cube. As a byproduct, we see that the product $\Phi\Psi$ vanishes on the vertices of this cube. Multiplying (1.3.7) and (1.3.8), we obtain

$$\Phi\Psi(z,w) = z^8 + 14z^4w^4 + w^8. \qquad (1.3.9)$$

The midpoints of the edges of any one of the tetrahedra are the vertices of the octahedron, the intersection of the two tetrahedra. Projected to $\hat{\mathbf{C}}$, these are

$$0, \infty, \pm 1, \pm i. \qquad (1.3.10)$$

Since z corresponds to 0, and w corresponds to ∞, a form Ω of degree 8 that vanishes on these vertices is:

$$\begin{aligned}
\Omega(z,w) &= zw(z^2 - w^2)(z^2 + w^2)\\
&= zw(z^4 - w^4). \qquad (1.3.11)
\end{aligned}$$

We say that the forms Φ, Ω, and Ψ *belong to the tetrahedron.* Recall that this means that $\Phi = 0$ on the vertices, $\Psi = 0$ on the midpoints of the edges, and $\Omega = 0$ on the centroids of the faces of the tetrahedron. We also see that the degrees of Φ, Ω, and Ψ are $|T|/\nu_0$, $|T|/\nu_1$, and $|T|/\nu_2$. The forms Φ and Ψ are invariants of the binary tetrahedral group T^*: $\chi_\Phi = \chi_\Psi$ with kernel D_2^*, and Ω is an absolute invariant of T^*. By the general discussion above, Φ^3, Ψ^3, and Ω^2 are linearly dependent. Comparing some of the coefficients in Φ^3, Ψ^3 and Ω^2, we have

$$\Phi^3 - 12\sqrt{3}i\Omega^2 - \Psi^3 = 0. \qquad (1.3.12)$$

It is once again convenient to make another observation about invariant forms. Given a G^*-invariant form ξ, the Hessian $Hess\,(\xi)$ defined by

$$Hess\,(\xi)(z,w) = \begin{vmatrix} \frac{\partial^2 \xi}{\partial z^2} & \frac{\partial^2 \xi}{\partial z \partial w} \\ \frac{\partial^2 \xi}{\partial w \partial z} & \frac{\partial^2 \xi}{\partial w^2} \end{vmatrix}$$

is also G^*-invariant. Furthermore, given two G^*-invariant forms ξ_0 and ξ_1, the Jacobian $Jac\,(\xi_0, \xi_1)$ defined by

$$Jac\,(\xi_0, \xi_1)(z,w) = \begin{vmatrix} \frac{\partial \xi_0}{\partial z} & \frac{\partial \xi_0}{\partial w} \\ \frac{\partial \xi_1}{\partial z} & \frac{\partial \xi_1}{\partial w} \end{vmatrix}$$

is also G^*-invariant. Thus, once we have a G^*-invariant form ξ_0, we automatically have two additional G^*-invariant forms: the Hessian $\xi_1 = Hess\,(\xi_0)$, and then the Jacobian $\xi_2 = Jac\,(\xi_0, \xi_1)$.

We first try this for the tetrahedral form Φ of degree 4. Since the Hessian $Hess\,(\Phi)$ is of degree 4, it can only be a linear combination of Φ and Ψ. By an easy computation, we see that $Hess\,(\Phi) = -48\sqrt{3}i\Psi$. (See Problem 1.26 listing all Hessians and Jacobians of Φ, Ψ, Ω.) The Jacobian of Φ and Ψ is of degree 6, so that it must be a constant multiple of Ω. Working out the top coefficient we see that $Jac\,(\Phi, \Psi) = 32\sqrt{3}i\Omega$.

Remark. For the dihedral forms α, β, and γ in (1.3.2)-(1.3.4), we have $Hess\,(\alpha) = -Hess\,(\beta) = -\frac{d^2(d-1)^2}{4}\gamma^{d-2}$, $Hess\,(\gamma) = 1$, and $Jac\,(\alpha, \beta) = \frac{d^2}{2}\gamma^{d-1}$, $Jac\,(\alpha, \gamma) = d\beta$, $Jac\,(\beta, \gamma) = d\alpha$.

Recall that the octahedral Möbius group contains the tetrahedral Möbius group. The same is true for the associated binary groups. Thus the O^*-invariant forms are polynomials of the tetrahedral forms Φ, Ψ, and Ω. To obtain the forms that belong to the octahedron, we first note that, by construction, Ω is the octahedral form that vanishes on the vertices of the octahedron. It is equally clear that the form $\Phi\Psi$ of degree 8 vanishes on the midpoints of the faces of the octahedron since these points are nothing but the vertices of the cubic dual. Note also that $Hess\,(\Omega) = -25\Phi\Psi$. We finally have to find a form of degree 12 that vanishes on the midpoints of the edges of the octahedron. Based on analogy with the tetrahedral forms, this form of degree 12 must be the Jacobian

$$Jac\,(\Omega, \Phi\Psi) = \Phi\, Jac\,(\Omega, \Psi) + \Psi\, Jac\,(\Omega, \Phi).$$

A simple computation shows that $Jac\,(\Omega, \Phi) = -4\Psi^2$ and $Jac\,(\Omega, \Psi) = -4\Phi^2$. Normalizing, we obtain that the middle octahedral form is

$$\frac{\Phi^3 + \Psi^3}{2}.$$

An explicit expression of this form is obtained by factoring

$$\frac{\Phi^3 + \Psi^3}{2} = \frac{1}{2}(\Phi + \Psi)(\Phi^2 - \Phi\Psi + \Psi^2)$$

$$
\begin{aligned}
&= \frac{1}{2}(\Phi + \Psi)((\Phi - \Psi)^2 + \Phi\Psi) \\
&= (z^4 + w^4)(z^8 - 34z^4w^4 + w^8) \\
&= z^{12} - 33z^8w^4 - 33z^4w^8 + w^{12}, \qquad (1.3.13)
\end{aligned}
$$

where we used (1.3.7)-(1.3.9). Summarizing, the three forms that belong to the octahedron are Ω, $(\Phi^3 + \Psi^3)/2$, and $\Phi\Psi$. They are all absolute invariants of the binary tetrahedral group T^*. As octahedral invariants, we have $\chi_\Omega = \chi_{(\Phi^3+\Psi^3)/2} = \pm 1$ with kernel $T^* \subset O^*$, while $\Phi\Psi$ is an absolute octahedral invariant. A linear relation among Ω^4, $(\Phi^3+\Psi^3)^2/4$, and $(\Phi\Psi)^3$ can be deduced from (1.3.12). Squaring both sides of the equation

$$\frac{\Phi^3 - \Psi^3}{2} = 6\sqrt{3}i\Omega^2,$$

we obtain

$$\frac{1}{4}(\Phi^3 - \Psi^3)^2 = \frac{1}{4}(\Phi^3 + \Psi^3)^2 - (\Phi\Psi)^3 = -108\Omega^4.$$

We thus have

$$108\Omega^4 + \left(\frac{\Phi^3 + \Psi^3}{2}\right)^2 - (\Phi\Psi)^3 = 0. \qquad (1.3.14)$$

Finally we turn to the case of the icosahedron. An icosahedral form $\mathcal{I}$ of degree 12 that vanishes on the projected vertices (1.2.16) is

$$
\begin{aligned}
\mathcal{I}(z,w) &= zw \prod_{j=0}^{4}(z - \omega^j(\omega + \omega^4)w) \prod_{j=0}^{4}(z - \omega^j(\omega^2 + \omega^3)w) \\
&= zw(z^5 - (\omega + \omega^4)^5 w^5)(z^5 - (\omega^2 + \omega^3)^5 w^5) \\
&= zw(z^5 - \tau^{-5}w^5)(z^5 + \tau^5 w^5) \\
&= zw(z^{10} + 11z^5w^5 - w^{10}). \qquad (1.3.15)
\end{aligned}
$$

Here we used (1.2.17) and the cyclotomic identity

$$\lambda^5 - 1 = \prod_{j=0}^{4}(\lambda - \omega^j)$$

with the substitutions $\lambda = \zeta/(\omega + \omega^4)$ and $\lambda = \zeta/(\omega^2 + \omega^3)$. In the last equality in (1.3.15) we used the relation $\tau^5 - \tau^{-5} = 11$. (To see this, first factor the right-hand side as

$$\tau^5 - \frac{1}{\tau^5} = \left(\tau - \frac{1}{\tau}\right)\left(\tau^4 + \tau^2 + 1 + \frac{1}{\tau^2} + \frac{1}{\tau^4}\right).$$

Squaring the defining relation $\tau - 1/\tau = 1$ we obtain $\tau^2 + 1/\tau^2 = 3$, and squaring again we get $\tau^4 + 1/\tau^4 = 7$. The relation follows.)

$\mathcal{I}$ is an absolute invariant of I^*. In fact, all invariants of I^* are absolute since I^* has no proper normal subgroup, and thereby no nontrivial charac-

ter (Problem 1.12). The Hessian $Hess\,(\mathcal{I})$ is an absolute invariant of degree 20. Thus, it must vanish on the centroids of the faces of the icosahedron. Normalizing, a straightforward computation gives

$$\begin{aligned}\mathcal{H}(z,w) &= \frac{1}{121}\,Hess\,(\mathcal{I})\,(z,w)\\ &= -(z^{20}+w^{20})\\ &\quad + 228(z^{15}w^5 - 494z^{10}w^{10} - z^5w^{15}).\end{aligned} \tag{1.3.16}$$

The Jacobian $Jac\,(\mathcal{I},\,\mathcal{H})$ is an absolute invariant of degree 30, so it must vanish on the midpoints of the edges of the icosahedron. Once again, direct computation gives

$$\begin{aligned}\mathcal{J}\,(z,w) &= -\frac{1}{20}\,Jac\,(\mathcal{I},\,\mathcal{H})(z,w)\\ &= (z^{30}+w^{30}) + 522(z^{25}w^5 - z^5w^{25})\\ &\quad - 10005(z^{20}w^{10} + z^{10}w^{20}).\end{aligned} \tag{1.3.17}$$

The icosahedral forms $\mathcal{I}$, $\mathcal{J}$, and $\mathcal{H}$ are algebraically dependent. A comparison of coefficients shows that

$$1728\mathcal{I}^5 - \mathcal{J}^2 - \mathcal{H}^3 = 0. \tag{1.3.18}$$

We now return to the general situation. Forming the six possible quotients of $\xi_0^{\nu_0}$, $\xi_1^{\nu_1}$, and $\xi_2^{\nu_2}$, we obtain six G-invariant rational functions that solve our problem. Since (1.3.6) is a linear relation among these forms, the six rational functions can be written in terms of each other in obvious ways. A more elegant way to express these is to use homogeneous coordinates and write

$$q : q-1 : 1 = -\lambda_2\xi_2^{\nu_2} : \lambda_1\xi_1^{\nu_1} : \lambda_0\xi_0^{\nu_0},$$

where $q = -\lambda_2\xi_2^{\nu_2}/(\lambda_0\xi_0^{\nu_0})$. We make an exception for the dihedron and define q such that $q : q-1 : 1 = \alpha^2 : \beta^2 : -\gamma^n$.

Remark. Klein called q the *fundamental rational function*, and the problem of inverting q the *fundamental problem.*

We summarize our results in the following tables:

Platonic solid	G	$\lvert G\rvert$	ν_0	ν_1	ν_2	ξ_0	ξ_1	ξ_2
dihedron	D_d	$2d$	2	2	d	α	β	γ
tetrahedron	T	12	3	2	3	Φ	Ω	Ψ
octahedron	O	24	4	2	3	Ω	$\frac{\Phi^3+\Psi^3}{2}$	$\Phi\Psi$
icosahedron	I	60	5	2	3	$\mathcal{I}$	$\mathcal{J}$	$\mathcal{H}$

G	$\lambda_0\xi_0^{\nu_0} + \lambda_1\xi_1^{\nu_1} + \lambda_2\xi_2^{\nu_2} = 0$	$q : q-1 : 1$
D_d	$\alpha^2 - \beta^2 + \gamma^d = 0$	$\alpha^2 : \beta^2 : -\gamma^d$
T	$\Phi^3 - 12\sqrt{3}i\Omega^2 - \Psi^3 = 0$	$\Psi^3 : -12\sqrt{3}i\Omega^2 : \Phi^3$
O	$108\Omega^4 + \left(\frac{\Phi^3+\Psi^3}{2}\right)^2 - (\Phi\Psi)^3 = 0$	$(\Phi\Psi)^3 : \left(\frac{\Phi^3+\Psi^3}{2}\right)^2 : 108\Omega^4$
I	$1728\mathcal{I}^5 - \mathcal{J}^2 - \mathcal{H}^3 = 0$	$\mathcal{H}^3 : -\mathcal{J}^2 : 1728\mathcal{I}^5$

We are now ready to prove that any finite Möbius group is conjugate to one of the Möbius groups listed in Section 1.2.

Theorem 1.3.1. *Any finite Möbius group $G \subset \mathcal{M}(\hat{\mathbf{C}})$ is cyclic, or conjugate to D_d, T, O, or I.*

PROOF. Let $G \subset \mathcal{M}(\hat{\mathbf{C}})$ be a finite subgroup. Let $a, b \in \mathbf{C}$ not on the same G-orbit. Consider the rational function $r : \mathbf{C} \to \mathbf{C}$ defined by

$$r(\zeta) = \prod_{g \in G} \frac{g(\zeta) - a}{g(\zeta) - b}, \quad \zeta \in \mathbf{C}.$$

The condition on a and b guarantees that r is nonconstant. Given $Z \in \mathbf{C}$, to find $\zeta \in \mathbf{C}$ such that $r(\zeta) = Z$, we need to solve the equation

$$\prod_{g \in G} (g(\zeta) - a) = Z \prod_{g \in G} (g(\zeta) - b).$$

Multiplying out the denominators in the linear fractions $g(\zeta)$, $g \in G$, this becomes a polynomial equation of degree $|G|$ with Z as a parameter. Extended to $\hat{\mathbf{C}}$, r becomes a holomorphic $|G|$-fold branched covering. On the other hand, r is also G-invariant. It follows that, for fixed $Z \in \hat{\mathbf{C}}$, the solution set of $r(\zeta) = Z$ is a single G-orbit, and $r : \hat{\mathbf{C}} \to \hat{\mathbf{C}}/G = \hat{\mathbf{C}}$ is the orbit map. $\zeta \in \hat{\mathbf{C}}$ is a branch point iff the G-orbit $G(\zeta)$ through ζ is not principal. In this case, $|G(\zeta)| = |G|/\nu$, and the branch number associated to ζ is $\nu - 1$. Letting Π denote the set of branch points of r, the *total branch number* is

$$B = \sum_{\Pi/G} \frac{|G|}{\nu}(\nu - 1),$$

where the summation is over all branch values. By the Riemann-Hurwitz relation (Problem 1.27), the total branch number is equal to $2|G| - 2$ since both the domain and the range have zero genera. This gives

$$2 - \frac{2}{|G|} = \sum_{\Pi/G} \left(1 - \frac{1}{\nu}\right).$$

Notice that this is the same diophantine restriction as (1.1.3). As before, the sum on the right-hand side either consists of two terms with $\nu_0 = \nu_1 = |G|$, or it consists of three tems with ν_0, ν_1, and ν_2 given in the table above. In the first case, let Z_0 and Z_1 be the branch values corresponding to ν_0 and ν_1. By performing a linear fractional transformation on the range, we may assume that $Z_0 = 0$ and $Z_1 = \infty$. For the rest of the cases, let Z_0, Z_1, and Z_2 be the three branch values. Performing again a linear fractional transformation on the range, we may assume that these are $Z_0 = 0$, $Z_1 = 1$, and $Z_2 = \infty$, and they correspond to ν_2, ν_1, and ν_0. (In the case of the dihedron ν_0 and ν_2 are switched.) We arrive at a scenario where the holomorphic branched coverings q and r have the *same branch points and branch*

numbers. By the well-known uniformization theorem for branched coverings (Farkas-Kra [1], pp. 219-220), the group G is conjugate to the Möbius group that defines q and the conjugation is a linear fractional transformation that establishes the conformal equivalence of the branched coverings q and r. The theorem follows.

Remark. Klein [1] gives an elementary argument to finish the proof of Theorem 1.3.1. For brevity, we relied on uniformization for branched coverings.

We complete this section by determining all absolute invariants for our finite Möbius groups. To formulate our results it will be convenient to introduce some notation. We let $\mathbf{C}[z, w]$ denote the ring of complex polynomials in the variables $z, w \in \mathbf{C}$. The group $SL(2, \mathbf{C})$ acts on $\mathbf{C}[z, w]$ in a natural way: $g \cdot \xi = \xi \circ g^{-1}$, $\xi \in \mathbf{C}[z, w]$, $g \in SL(2, \mathbf{C})$. This makes $\mathbf{C}[z, w]$ an $SL(2, \mathbf{C})$-*module*, a representation space for the group $SL(2, \mathbf{C})$. The linear subspace $W_p \subset \mathbf{C}[z, w]$ of degree p homogeneous polynomials is $SL(2, \mathbf{C})$-invariant, a (finite dimensional) $SL(2, \mathbf{C})$-module itself. We have the direct sum decomposition

$$\mathbf{C}[z, w] = \sum_{p=0}^{\infty} W_p.$$

For any finite subgroup $G^* \subset SL(2, \mathbf{C})$, the fixed point set $W_p^{G^*}$ is the linear space of absolute invariants of degree p for G^*. We call $\mathbf{C}[z, w]^{G^*}$ the *ring of absolute invariants* for G^*. The case of the binary icosahedral group $G^* = I^*$ is particularly simple because we saw that any invariant for I^* is automatically absolute. We also proved that if ξ is an I^* invariant form then ξ can be written as a polynomial of the basic invariants $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{H}$. We obtain that ξ can be viewed as an element of the polynomial ring $\mathbf{C}[\mathcal{I}, \mathcal{J}, \mathcal{H}]$. The representation of ξ as a polynomial in $\mathcal{I}$, $\mathcal{J}$, $\mathcal{H}$ is not unique, however, since the the basic invariants are algebraically dependent. This dependence is expressed in the relation (1.3.18). To make the representation of ξ in terms of the basic invariants unique, we factor the ring $\mathbf{C}[\mathcal{I}, \mathcal{J}, \mathcal{H}]$ by the principal ideal generated by $1728\mathcal{I}^5 - \mathcal{J}^2 - \mathcal{H}^3$. We find that the ring of absolute invariants for I^* is

$$\mathbf{C}[z, w]^{I^*} = \mathbf{C}[\mathcal{I}, \mathcal{J}, \mathcal{H}]/(1728\mathcal{I}^5 - \mathcal{J}^2 - \mathcal{H}^3). \tag{1.3.19}$$

By Theorem 1.3.1, it remains for us to treat the cases when G is cyclic, or is one of the binary groups D_d^*, T^*, O^*.

Let $G = C_d$ be generated by $\epsilon = e^{i\frac{2\pi}{d}}$ viewed as a quaternion. (Here we are interested in forms rather than the associated rational function and have changed the notation accordingly.) Since ϵ generates C_d, the condition of absolute C_d-invariance of a form ξ reduces to

$$\xi(\epsilon z, \epsilon^{-1} w) = \xi(z, w), \quad z, w \in \mathbf{C}.$$

It is clear that the dihedral form γ defined in (1.3.4) is an absolute invariant. Since any monomial ξ that contains both variables z and w is divisible by γ, we are left with the case where ξ is a power of z or w. Since the least power that makes ξ an absolute invariant is d, we see that the ring of absolute invariants $\mathbf{C}[z,w]^{C_d}$ is generated by z^d, w^d, and zw. Using the dihedral invariants α and β, we thus have

$$\mathbf{C}[z,w]^{C_d} = \mathbf{C}[\alpha+\beta, \alpha-\beta, \gamma]/((\alpha+\beta)(\alpha-\beta)-\gamma^n),$$

or equivalently,

$$\mathbf{C}[z,w]^{C_d} = \mathbf{C}[z^d, w^d, zw]/(z^d w^d - (zw)^d). \tag{1.3.20}$$

Next, let $G = D_d^*$. Since $C_d^* \subset D_d^*$, the absolute invariants of D_d^* are polynomials in the absolute invariants for $C_d^* = C_{2d}$, i.e. they are polynomials in $(\alpha+\beta)^2$, $(\alpha-\beta)^2$, and γ. A simple computation gives the generators:

$$(\alpha+\beta)^2 + (\alpha-\beta)^2,\ \gamma((\alpha+\beta)^2 - (\alpha-\beta)^2),\ \gamma^2,$$

or equivalently,

$$z^{2d} + w^{2d},\ zw(z^{2d} - w^{2d}),\ z^2 w^2.$$

The obvious relation between these generators is

$$[z^{2d} + w^{2d}]^2 z^2 w^2 - [zw(z^{2d} - w^{2d})]^2 - 4[z^2 w^2]^{d+1} = 0.$$

We obtain

$$\begin{aligned}\mathbf{C}[z,w]^{D_d^*} = \mathbf{C}[&z^{2d} + w^{2d}, zw(z^{2d} - w^{2d}), z^2 w^2]/ \\ &([z^{2d} + w^{2d}]^2 z^2 w^2 - [zw(z^{2d} - w^{2d})]^2 - 4[z^2 w^2]^{d+1}).\end{aligned} \tag{1.3.21}$$

As noted above, Ω, $(\Phi^3 + \Psi^3)/2$, and $\Phi\Psi$ are absolute invariants of the binary tetrahedral group T^*. A simple computation involving the characters χ_Φ and χ_Ψ shows that they generate $\mathbf{C}[z,w]^{T^*}$. We have

$$\mathbf{C}[z,w]^{T^*} = \mathbf{C}\left[\Omega, \frac{\Phi^3+\Psi^3}{2}, \Phi\Psi\right] \Big/ \left(108\Omega^4 + \left(\frac{\Phi^3+\Psi^3}{2}\right)^2 - (\Phi\Psi)^3\right). \tag{1.3.22}$$

The binary tetrahedral group T^* is a subgroup of the binary octahedral group O^*, so that the absolute invariants of O^* are polynomials in Ω, $(\Phi^3 + \Psi^3)/2$, and $\Phi\Psi$. We have $\chi_\Omega = \chi_{(\Phi^3+\Psi^3)/2} = \pm 1$, and $\Phi\Psi$ is an absolute invariant. A set of generators for $\mathbf{C}[z,w]^{O^*}$ is

$$\Omega^2,\ \Omega\frac{\Phi^3+\Psi^3}{2},\ \Phi\Psi.$$

Multiplying both sides of (1.3.14) by Ω^2, we obtain the algebraic relation between these generators:

$$108[\Omega^2]^3 + \left[\Omega\frac{\Phi^3+\Psi^3}{2}\right]^2 - \Omega^2[\Phi\Psi]^3 = 0.$$

We have

$$\mathbf{C}[z,w]^{O^*} = \mathbf{C}\left[\Omega^2, \Omega\frac{\Phi^3+\Psi^3}{2}, \Phi\Psi\right] \Big/ \left(108[\Omega^2]^3 + \left[\Omega\frac{\Phi^3+\Psi^3}{2}\right]^2 - \Omega^2[\Phi\Psi]^3\right). \tag{1.3.23}$$

Remark. In Klein's theory of the icosahedron the absolute invariants play a crucial role as they solve what he calls the *form problem.* In fact, he showed that the inverse q^{-1} of the fundamental rational function q can be written as the ratio of the inverses of absolute G^*-invariants. In our approach, it was convenient to express all absolute tetrahedral and octahedral invariants as polynomials in Φ, Ω, and Ψ.

We tabulate the results of this section as follows:

G^*	Absolute Invariants			Relation
D_d^*	$z^{2d}+w^{2d}$	$zw(z^{2d}-w^{2d})$	z^2w^2	$[z^{2d}+w^{2d}]^2z^2w^2 - [zw(z^{2d}-w^{2d})]^2 = 4[z^2w^2]^{d+1}$
T^*	Ω	$\frac{\Phi^3+\Psi^3}{2}$	$\Phi\Psi$	$108\Omega^4 + \left(\frac{\Phi^3+\Psi^3}{2}\right)^2 - (\Phi\Psi)^3 = 0$
O^*	Ω^2	$\Omega\frac{\Phi^3+\Psi^3}{2}$	$\Phi\Psi$	$108[\Omega^2]^3 + \left[\Omega\frac{\Phi^3+\Psi^3}{2}\right]^2 - \Omega^2[\Phi\Psi]^3 = 0$
I^*	$\mathcal{I}$	$\mathcal{J}$	$\mathcal{H}$	$1728\mathcal{I}^5 - \mathcal{J}^2 - \mathcal{H}^3 = 0$

1.4 Minimal Immersions of the 3-sphere into Spheres

In this section we introduce the "equivariant construction." This enables us to manufacture a variety of minimal immersions of S^3 into spheres. The immersions will be realized as orbit maps under the action of $SU(2) = S^3$ on (submodules of) the $SL(2,\mathbf{C})$-module $\mathbf{C}[z,w]$. The equivariant construction was used by Mashimo [1] to obtain a minimal immersion $f : S^3 \to S^6$ of degree 6, a minimum codimension example. (For a very general formulation of Mashimo's result, see Weingart [1], Section 3.2.) Here we follow the treatment of DeTurck and Ziller [1,2]. We will use some basic results in the representation theory of $SU(2)$. Standard references are Börner [1], Knapp [1], and Vilenkin [1].

As in the previous section, we let $W_p \subset \mathbf{C}[z,w]$, $p \geq 0$, denote the $SU(2)$-module of complex homogeneous polynomials of degree p in the

complex variables z and w. We have $\dim_{\mathbf{C}} W_p = p+1$. The standard basis in W_p is $\{z^{p-q}w^q\}_{q=0}^{p}$. It is well-known that W_p is irreducible as a complex $SU(2)$-module (a complex representation space for $SU(2)$ with no proper invariant subspace). In fact, a complex irreducible $SU(2)$-module W with $\dim_{\mathbf{C}} W = p+1$ is equivalent to W_p. We see that

$$\mathbf{C}[z,w] = \sum_{p=0}^{\infty} W_p.$$

is actually the decomposition of the complex $SU(2)$-module $\mathbf{C}[z,w]$ into irreducible components. For $p = 2d$ even, consider the complex anti-linear self-map of W_{2d} defined by the requirement that it should send $z^q w^{2d-q}$ to $(-1)^q z^{2d-q} w^q$, $q = 0, \ldots, 2d$. The fixed point set of this map is an irreducible *real* $SU(2)$-submodule R_{2d} of W_{2d}. The complexification of R_{2d} is W_{2d}. The standard basis in R_{2d} is given by

$$\{z^{2d-l}w^l + (-1)^l z^l w^{2d-l}, i(z^{2d-l}w^l - (-1)^l z^l w^{2d-l})\}_{l=0}^{d-1}$$
$$\cup \{i^d z^d w^d\} \subset R_{2d}.$$

When p is odd, W_p, as a real $SU(2)$-module, is irreducible.

We now turn to the equivariant construction for the $SU(2)$-module W_p. (For simplicity, we discuss only the irreducible case. The equivariant construction for reducible $SU(2)$-modules can be treated analogously, and they will appear in examples in later chapters.) Consider a nonzero polynomial $\xi \in W_p$, $p \geq 2$. In terms of the standard basis in W_p, we write

$$\xi(z,w) = \sum_{q=0}^{p} c_q z^{p-q} w^q, \quad c_q \in \mathbf{C},\ q = 0, \ldots, p. \tag{1.4.1}$$

Let $f_\xi : S^3 \to W_p$ be the orbit map through ξ:

$$f_\xi(g) = g \cdot \xi = \xi \circ L_{g^{-1}}, \quad g \in SU(2) = S^3,$$

where L stands for left quaternionic multiplication. More explicitly, setting $g = a + jb \in S^3$, $a, b \in \mathbf{C}$, we have

$$f_\xi(a+jb)(z,w) = \xi(\bar{a}z + \bar{b}w, -bz + aw), \quad z, w \in \mathbf{C}. \tag{1.4.2}$$

The definition of f_ξ automatically extends to $\mathbf{H} = \mathbf{R}^4 (\supset S^3)$, and we see that the components of f_ξ (relative to, say, the standard basis) are homogeneous degree p polynomials in the *real* variables $\Re(a)$, $\Im(a)$, $\Re(b)$, $\Im(b)$.

Being an orbit map, f_ξ is automatically $SU(2)$-equivariant:

$$f_\xi \circ L_g = g \cdot f_\xi, \ \ g \in SU(2).$$

Indeed, for $g, g' \in SU(2)$, we have

$$\begin{aligned}(f_\xi \circ L_g)(g') &= f_\xi(gg') = \xi \circ L_{(gg')^{-1}} \\ &= \xi \circ L_{(g')^{-1}} \circ L_{g^{-1}} = g \cdot f_\xi(g').\end{aligned}$$

We endow W_p with an $SU(2)$-invariant scalar product. (A specific choice of the scalar product will be given below.) Since $SU(2)$ acts transitively on S^3, the image of f_ξ is contained in a sphere of W_p. If ξ has unit length, the image is contained in the unit sphere $S_{W_p} = S^{2p+1}$ of W_p. Restricting, we obtain a map $f_\xi : S^3 \to S^{2p+1}$.

Applying the complex form of the Laplacian

$$\triangle = 4\left(\frac{\partial^2}{\partial a \partial \bar{a}} + \frac{\partial^2}{\partial b \partial \bar{b}}\right),$$

to both sides of (1.4.2), an easy computation in the chain rule shows that the components of f_ξ are harmonic (Problem 1.28).

As will be discussed in Section 2.1, a harmonic degree p homogeneous polynomial on $\mathbf{R}^{m+1}$ restricts to an eigenfunction of the spherical Laplacian $\triangle^{S^m}$ on $S^m \subset \mathbf{R}^{m+1}$ with eigenvalue $p(p+m-1)$. (This is based on comparing the Euclidean and spherical Laplacians.) In our case ($m = 3$) we thus have

$$\triangle^{S^3} f_\xi = p(p+2) f_\xi,$$

as vector valued functions. In general, a map $f : S^m \to S_V$ into the unit sphere S_V of a Euclidean vector space V is a *p-eigenmap* if

$$\triangle^{S^m} f = p(p+m-1) f.$$

Chapter 2 will be devoted to the study of eigenmaps. Comparing the two equations above, we see that any orbit map $f_\xi : S^3 \to S_{W_p}$ is a p-eigenmap.

To work out the equivariant construction in specific examples, we need a convenient scalar product on $\mathcal{H}^p$, the space of complex valued harmonic degree p homogeneous polynomials on $\mathbf{R}^4 = \mathbf{C}^2$. In general, given $\chi_1, \chi_2 \in \mathcal{H}^p$ written in terms of the variables $z, \bar{z}, w, \bar{w}$, we define

$$\langle \chi_1, \chi_2 \rangle_{\mathcal{H}^p} = \Re\left[\chi_1\left(\frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial \bar{w}}, \frac{\partial}{\partial z}, \frac{\partial}{\partial w}\right)\bar{\chi}_2\right], \tag{1.4.3}$$

where χ_1 *acts on* $\bar{\chi}_2$ *as a polynomial differential operator.*

An equivalent form of this scalar product is

$$\langle \chi_1, \chi_2 \rangle_{\mathcal{H}^p} = \frac{1}{4^p p!} \triangle^p \Re(\chi_1 \bar{\chi}_2), \quad \chi_1, \chi_2 \in \mathcal{H}^p. \tag{1.4.4}$$

The equivalence of (1.4.3) and (1.4.4) can be seen by working out the right-hand sides on monomials.

Although less convenient in computations, (1.4.4) shows that this scalar product is *invariant* under the action of the orthogonal group $SO(4)$ on $\mathcal{H}^p$, where the latter is given by $g \cdot \chi = \chi \circ g^{-1}$, $\chi \in \mathcal{H}^p$, $g \in SO(4)$. (This is because the Laplacian commutes with isometries.)

Remark. Up to a (real) constant multiple, (1.4.3) is the L^2-scalar product defined by integration on $S^3 \subset \mathbf{C}^2$. (For the explicit form of (1.4.3) in terms of the L^2-scalar product, see Problem 2.2.) This follows from the

fact that $\mathcal{H}^p$ is irreducible as an $SO(4)$-module, and both the L^2-scalar product and (1.4.3) are $SO(4)$-invariant. (Indeed, the discrepancy between two scalar products is made up by a hermitian linear endomorphism A of $\mathcal{H}^p$; cf. Problem 1.30. Since both scalar products are $SO(4)$-invariant, A is an $SO(4)$-module endomorphism. By Schur's lemma, irreducibility of $\mathcal{H}^p$ as an $SO(4)$-module implies that A must be a complex constant multiple of the identity. Since A is hermitian symmetric, the constant is real.)

In view of the scalar product (1.4.3), we scale the elements of the standard (orthogonal) basis in W_p so that the new basis becomes orthonormal:

$$\left\{\frac{1}{\sqrt{(p-q)!q!}}z^{p-q}w^q\right\}_{q=0}^{p} \subset W_p. \tag{1.4.5}$$

In a similar vein, for $p = 2d$, the elements of the standard orthonormal basis in R_{2d} are

$$\begin{aligned}
\xi_{2l+1} &= \frac{1}{\sqrt{2(2d-l)!l!}}(z^{2d-l}w^l + (-1)^l z^l w^{2d-l}), \quad l = 0, \ldots, d-1 \\
\xi_{2l+2} &= \frac{i}{\sqrt{2(2d-l)!l!}}(z^{2d-l}w^l - (-1)^l z^l w^{2d-l}), \quad l = 0, \ldots, d-1 \\
\xi_{2d+1} &= \frac{i^d}{d!}z^d w^d.
\end{aligned} \tag{1.4.6}$$

With these orthonormal bases, we have the following identifications for the unit spheres: $S_{W_p} = S^{2p+1}$ and $S_{R_{2d}} = S^{2d}$.

We now let $p = 2$, and work out the orbit map f_ξ for W_2 explicitly. As noted above, for each $\xi \in W_2$ of unit length, f_ξ is a quadratic (2-)eigenmap.

A typical element in W_2 is

$$\xi(z, w) = c_0 z^2 + c_1 zw + c_2 w^2, \quad z, w \in \mathbf{C}.$$

Using the scalar product (1.4.3), the assumption that ξ has unit length in W_2 reduces to

$$2|c_0|^2 + |c_1|^2 + 2|c_2|^2 = 1. \tag{1.4.7}$$

By (1.4.2), the orbit map $f_\xi : S^3 \to S_{W_2}$ has the form

$$\begin{aligned}
f_\xi(a+jb)(z,w) &= \xi(\bar{a}z + \bar{b}w, -bz + aw) \\
&= c_0(\bar{a}z + \bar{b}w)^2 + c_1(\bar{a}z + \bar{b}w)(-bz + aw) \\
&\quad + c_2(-bz + aw)^2 \\
&= (c_0\bar{a}^2 - c_1\bar{a}b + c_2b^2)z^2 \\
&\quad + (2c_0\bar{a}\bar{b} + c_1(|a|^2 - |b|^2) - 2c_2ab)zw \\
&\quad + (c_0\bar{b}^2 + c_1a\bar{b} + c_2a^2)w^2.
\end{aligned} \tag{1.4.8}$$

With respect to the orthonormal basis (1.4.5), $f_\xi(z,w) = f_\xi(z+jw)$ can be written *in components* as

$$f_\xi(z,w) = \Big(\sqrt{2}(c_0\bar{z}^2 - c_1\bar{z}w + c_2w^2),\\ 2c_0\bar{z}\bar{w} + c_1(|z|^2 - |w|^2) - 2c_2zw,\\ \sqrt{2}(c_0\bar{w}^2 + c_1z\bar{w} + c_2z^2)\Big),$$

where we switched back from $a, b \in \mathbf{C}$ to $z, w \in \mathbf{C}$.

Example 1.4.1. In (1.4.7) we set $c_0 = c_1 = 0$ and $c_2 = 1/\sqrt{2}$. Then, up to a permutation and a sign change of the components whose net effect is an isometry on the range, the corresponding quadratic eigenmap f_ξ is the *complex Veronese map* $Ver^{\mathbf{C}} : S^3 \to S^5$ given by

$$Ver^{\mathbf{C}}(z,w) = (z^2, \sqrt{2}zw, w^2).$$

The complex Veronese map defines a holomorphic imbedding of $\mathbf{C}P^1$ into the complex projective plane $\mathbf{C}P^2$. In terms of homogeneous coordinates, $[X : Y : Z]$ in $\mathbf{C}P^2$, the image is the complex quadric $Y^2 = 2XZ$. This is because, for $X = z^2$, $Y = \sqrt{2}zw$, $Z = w^2$, we have

$$Y^2 = (\sqrt{2}zw)^2 = 2z^2w^2 = 2XZ.$$

We now restrict ξ to the real $SU(2)$-submodule $R_2 \subset W_2$. This means that we assume that $c_0 = \bar{c}_2$ and c_1 is purely imaginary. The condition of sphericality (1.4.7) reduces to

$$|c_1|^2 + 4|c_2|^2 = 1,\ c_1 \in i\mathbf{R},\ c_2 \in \mathbf{C}. \tag{1.4.9}$$

The elements of the orthonormal basis (1.4.6) for R_2 are

$$\xi_1 = \frac{1}{2}(z^2 + w^2), \quad \xi_2 = \frac{1}{2}(z^2 - w^2), \quad \xi_3 = izw.$$

Eliminating c_0 in (1.4.8) and substituting $z^2 = \xi_1 + \xi_2$ and $w^2 = \xi_1 - \xi_2$, a simple computation gives $f_\xi : S^3 \to S^2$ (in components) as

$$f_\xi(z,w) = \Big(2\Re(c_2(z^2 + w^2) + c_1z\bar{w}),\\ -2i\Im(c_2(z^2 - w^2) + c_1z\bar{w}),\\ -ic_1(|z|^2 - |w|^2) + 4\Im(c_2zw)\Big),$$

where we switched back to $z, w \in \mathbf{C}$ again. Since $\Re(\bar{c}_2\bar{w}^2) = \Re(c_2w^2)$ and $\Im(\bar{c}_2\bar{w}^2) = -\Im(c_2w^2)$, up to an isometry on the range, this rewrites as

$$f_\xi(z,w) = \big(2(c_1z\bar{w} + c_2z^2 + \bar{c}_2\bar{w}^2), -ic_1(|z|^2 - |w|^2) + 4\Im(c_2zw)\big). \tag{1.4.10}$$

Here the first component is complex valued, the second component is real valued (since c_1 is purely imaginary), and we identified $\mathbf{C} \times \mathbf{R}$ with $\mathbf{R}^3$.

Example 1.4.2. Let $c_0 = c_2 = 0$ and $c_1 = i$. Up to an isometry on the range, the corresponding quadratic eigenmap f_ξ is the Hopf map $Hopf : S^3 \to S^2$ given by

$$Hopf(z,w) = (|z|^2 - |w|^2, 2z\bar{w}).$$

Being an orbit map, $Hopf : S^3 \to S^2$ is $SU(2)$-equivariant. The equivariance is with respect to a homomorphism $\rho_{Hopf} : SU(2) \to SO(3)$ that (for reasons of dimension) is onto (Problem 1.31).

To generalize Example 1.4.2, we set $\xi(z,w) = c_d z^d w^d$, $d \geq 1$. If we let $c_d = i^d/d!$ then ξ is of unit length and is contained in R_{2d}. We work out the $(2d)$-eigenmap $f : S^3 \to S^{2d}$. We have

$$\begin{aligned} f_\xi(a+jb)(z,w) &= c_d(\bar{a}z + \bar{b}w)^d(-bz+aw)^d \\ &= c_d\Bigg((|a|^2 - |b|^2)zw \\ &\quad - 2\Re(a\bar{b})\frac{z^2-w^2}{2} + 2i\Im(a\bar{b})\frac{z^2+w^2}{2}\Bigg)^d. \end{aligned}$$

We recognize that the coefficients (depending on $a, b \in \mathbf{C}$) are the components of the Hopf map:

$$Hopf^0(a,b) = |a|^2 - |b|^2, \ Hopf^1(a,b) = 2\Re(a\bar{b}), \ Hopf^2(a,b) = 2\Im(a\bar{b}).$$

This means that $f_\xi : S^3 \to S^{2d}$ factors through the Hopf map. The factor is called the generalized *Veronese map* $Ver_d : S^2 \to S^{2d}$ of degree d. Replacing the components of the Hopf map by x_0, x_1 and x_2, the factor is given by

$$Ver_d(x) = c_d\left(x_0 zw - x_1\frac{z^2-w^2}{2} + ix_2\frac{z^2+w^2}{2}\right)^d, \tag{1.4.11}$$

where $x = (x_0, x_1, x_2) \in S^2 \subset \mathbf{R}^3$. The components of Ver_d are homogeneous harmonic polynomials of degree d, so that $Ver_d : S^2 \to S^{2d}$ is a d-eigenmap. Harmonicity follows since

$$\begin{aligned} \triangle Ver_d &= \left(\frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) Ver_d \\ &= d(d-1)\left((zw)^2 + \left(\frac{z^2-w^2}{2}\right)^2 - \left(\frac{z^2-w^2}{2}\right)^2\right) Ver_d = 0. \end{aligned}$$

As noted above, the Hopf map is $SU(2)$-equivariant with respect to the homomorphism $\rho_{Hopf} : SU(2) \to SO(3)$. Thus, Ver_d is $SO(3)$-equivariant with respect to a homomorphism $\rho_d : SO(3) \to SO(2d+1)$. $SO(3)$ acts transitively on S^2. In addition, at any point $x \in S^2$, the isotropy subgroup $SO(3)_x$ is a circle group with typical element $R_{\theta,x}$, the rotation with angle θ and axis $\mathbf{R}\cdot x$ (Theorem 1.1.1 and Section 1.2). $SO(3)_x$ acts on the tangent space $T_x(S^2)$ at x via rotations. Since Ver_d is equivariant, this means that

the differential of Ver_d at x is, up to a constant multiple $c(x)$, a linear isometric imbedding of $T_x(S^2)$ into $T_{Ver_d(x)}(S^{2d})$.

We say that Ver_d is *conformal* with conformality factor $c : S^2 \to \mathbf{R}$. Since $SO(3)$ is transitive on S^2 and Ver_d is $SO(3)$-equivariant, c is constant. In general, a map $f : S^m \to S_V$ is said to be conformal if

$$\langle f_*(X), f_*(Y)\rangle = c(x)\langle X, Y\rangle, \quad X, Y \in T_x(S^m),\ x \in S^m,$$

where $c : S^m \to \mathbf{R}$ is the conformality factor.

It is a general fact which will be proved in Section 3.1 (Proposition 3.1.1) that a conformal p-eigenmap $f : S^m \to S_V$ is an isometric minimal immersion with induced metric $m/(p(p+m-1))$ times the original metric on the domain. In particular, keeping the original metric on the domain, the conformality factor is $c = p(p+m-1)/m$. (Some basic definitions and facts about minimal immersions are assembled in Appendix 2.) Since the components of a p-eigenmap are restrictions of harmonic degree p homogeneous polynomials to S^m, we say that a conformal p-eigenmap is a *minimal immersion of degree p*. As an example, we see that $Ver_d : S^2 \to S^{2d}$ is a minimal immersion of degree d.

As concluded above, the components of Ver_d are (restrictions of) harmonic degree d homogeneous polynomials on $\mathbf{R}^3$. Thus, for d even, Ver_d factors through the antipodal map giving a minimal immersion of the real projective plane $\mathbf{R}P^2$ into S^{2d}. Inspecting the components of Ver_d, we see that this latter immersion is actually a (minimal) imbedding of $\mathbf{R}P^2$ into S^{2d}. In a similar vein, for d odd, Ver_d is a minimal imbedding of S^2 into S^{2d}.

Remark 1. The components of the Veronese map $Ver_d : S^2 \to S^{2d}$ are orthogonal with the same norm. This can either be checked by explicit computation (see Problem 2.3), or follows from Schur's orthogonality relations (Problem 1.29), or it will follow from the rigidity results (Corollary 2.5.2) to be found in Section 2.5.

Remark 2. We derived here a somewhat unusual explicit form of the Veronese immersions $Ver_d : S^2 \to S^{2d}$. These will play a primary role in Section 2.5. Note that harmonicity of Ver_d also follows from a general composition formula (since the Hopf map is a Riemannian submersion, cf. Eells-Lemaire [1]) but we preferred to give a direct proof.

For $d = 2$, we determine the Veronese map $Ver = Ver_2 : S^2 \to S^4$ explicitly. A simple computation gives

$$\begin{aligned} Ver\,(x) = c_2\Big(&\frac{x_1^2 - x_2^2}{4}(z^4 + w^4) - i\frac{x_1x_2}{2}(z^4 - w^4) \\ &- x_0x_1(z^3w - zw^3) + ix_0x_2(z^3w + zw^3) \\ &+ (2x_0^2 - x_1^2 - x_2^2)\frac{z^2w^2}{2}\Big). \end{aligned}$$

Up to constant multiples, the factors involving $z, w \in \mathbf{C}$ are the five elements of the orthonormal basis (1.4.6) in R_4 ($d = 2$). Thus, the Veronese map in components is

$$Ver\,(x) = c_2(\sqrt{3}(x_1^2 - x_2^2), -2\sqrt{3}x_1x_2, -2\sqrt{3}x_0x_1, 2\sqrt{3}x_0x_2, \\ 2x_0^2 - x_1^2 - x_2^2).$$

As noted above, the image of *Ver* is $\mathbf{R}P^2$ minimally imbedded in S^4. This image is called the *Veronese surface*. Figure 1 shows various stereographic projections of this surface.

We now return to the general setting. As noted above, in Section 3.1 we will prove that a conformal p-eigenmap is a minimal immersion of degree p. Being a p-eigenmap, conformality of the orbit map f_ξ means that the differential $(f_\xi)_*$ satisfies

$$\langle (f_\xi)_*(X), (f_\xi)_*(Y)\rangle = \frac{p(p+2)}{3}\langle X, Y\rangle, \quad X, Y \in T_x(S^3),\ x \in S^3,$$

where we inserted the actual conformality factor $c = p(p+2)/3$. Since $SU(2)$ acts transitively on S^3, by $SU(2)$-equivariance of f_ξ, conformality holds iff f_ξ is homothetic (its differential is a constant multiple of an isometry) at a point, say, $1 \in S^3$. To obtain minimal immersions as orbit maps we thus have to work out the condition of homothety of f_ξ at $1 \in S^3$. To do this, we identify the tangent space $T_1(S^3) = T_1(SU(2))$ with the Lie algebra $su(2)$. An orthonormal basis in $su(2)$ is given by

$$Z = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}. \tag{1.4.12}$$

Let $\xi \in W_p$, $p \geq 4$. The condition of homothety gives a set of constraints on the coefficients c_q, $q = 0, \ldots, p$, of the expansion (1.4.1) of ξ. We derive these constraints in the remainder of this section.

The 1-parameter subgroup in $S^3 = SU(2)$ corresponding to $Z \in su(2)$ is $t \mapsto e^{it}$, $t \in \mathbf{R}$. This is because e^{it} acts on $z + jw \in S^3$ by left quaternionic multiplication as

$$e^{it}(z + jw) = e^{it}z + je^{-it}w,$$

and, differentiating at $t = 0$, we obtain that $d/dt|_{t=0}e^{it} \in su(2)$ is a diagonal matrix with diagonal elements i and $-i$, and this is Z. Using (1.4.2) we now compute

$$\begin{aligned}(f_\xi)_*(Z) &= \frac{d}{dt} f_\xi(e^{it})|_{t=0}(z, w) \\ &= \frac{d}{dt}\xi(e^{-it}z, e^{it}w)|_{t=0} \\ &= \sum_{q=0}^{p} c_q \frac{d}{dt} e^{i(2q-p)t}|_{t=0} z^{p-q}w^q\end{aligned}$$

$$= i\sum_{q=0}^{p}(2q-p)c_q z^{p-q}w^q.$$

Similarly, the 1-parameter subgroup in $S^3 = SU(2)$ corresponding to $X \in su(2)$ is $t \mapsto \cos t - j\sin t$, $t \in \mathbf{R}$. This is because

$$(\cos t - j\sin t)(z + jw) = (\cos tz + \sin tw) + j(\cos tw - \sin tz)$$

and, differentiating at $t = 0$, this becomes $w - jz$.

Using this and (1.4.2), we have

$$\begin{aligned}(f_\xi)_*(X) &= \frac{d}{dt} f_\xi(\cos t - j\sin t)|_{t=0}(z,w)\\ &= \frac{d}{dt}\xi(z\cos t - w\sin t, z\sin t + w\cos t)|_{t=0}\\ &= \sum_{q=0}^{p} c_q \frac{d}{dt}(z\cos t - w\sin t)^{p-q}(z\sin t + w\cos t)^q|_{t=0}\\ &= -\sum_{q=0}^{p-1}(p-q)c_q z^{p-q-1}w^{q+1} + \sum_{q=1}^{p} qc_q z^{p-q+1}w^{q-1}.\end{aligned}$$

Finally, by similar computations

$$(f_\xi)_*(Y) = -i\left(\sum_{q=0}^{p-1}(p-q)c_q z^{p-q-1}w^{q+1} + \sum_{q=1}^{p} qc_q z^{p-q+1}w^{q-1}\right).$$

We evaluate all possible scalar products of these with respect to (1.4.3). We find that f_ξ is homothetic at $1 \in S^3$ iff the following system of equations is satisfied:

$$\begin{aligned}&\sum_{q=0}^{p}(p-q)!q!|c_q|^2 = 1,\\ &\sum_{q=0}^{p}(2q-p)^2(p-q)!q!|c_q|^2 = \frac{p(p+2)}{3},\\ &\sum_{q=0}^{p-2}(q+2)!(p-q)!c_q\bar{c}_{q+2} = 0,\\ &\sum_{q=0}^{p-1}(p-2q-1)(q+1)!(p-q)!c_q\bar{c}_{q+1} = 0.\end{aligned} \tag{1.4.13}$$

Indeed, the first equation expresses the fact that $|\xi|^2 = 1$. The second equation is equivalent to $|(f_\xi)_*(Z)|^2 = p(p+2)/3$. (As noted above, the conformality factor for f_ξ is necessarily $p(p+2)/3$.) Using the first two equations, the real part of the third equation is equivalent to $|(f_\xi)_*(X)|^2 =$

$p(p+2)/3$, etc. Altogether, we have 6 constraints on the $2(p+1)$ real variables $\Re(c_q)$ and $\Im(c_q)$, $q = 0, \ldots, p$.

For $p = 2d$ even, to reduce the range dimension, we require $\xi \in R_{2d}$ so that f_ξ will map into R_{2d}. Using the definition of $R_{2d} \subset W_{2d}$, we obtain that this additional requirement for ξ translates into

$$\begin{aligned} c_{2d} = \bar{c}_0, \;\; c_{2d-1} = -\bar{c}_1, \ldots, c_{d+1} = (-1)^{d+1}\bar{c}_{d-1}, \\ c_d = i^d r, \;\; r \in \mathbf{R}. \end{aligned} \tag{1.4.14}$$

These give $2d+1$ additional constraints. With these the polynomial ξ in (1.4.1) reduces to

$$\xi = c_0 z^{2d} + \bar{c}_0 w^{2d} + c_1 z^{2d-1} w - \bar{c}_1 z w^{2d-1} + \ldots + i^d r z^d w^d. \tag{1.4.15}$$

Using the constraints in (1.4.14) again, the system of equations (1.4.13) for R_{2d}, expressing that $f_\xi : S^3 \to S^{2d}$ is a minimal immersion of degree $2d$, reduces to the following

$$\begin{aligned} &\sum_{q=0}^{d-1} 2(2d-q)!q!|c_q|^2 + (d!)^2 r^2 = 1, \\ &+\sum_{q=0}^{d-1} 2(2q-2d)^2(2d-q)!q!|c_q|^2 = \frac{4d(d+1)}{3}, \\ &\sum_{q=0}^{d-3} 2(q+2)!(2d-q)!c_q\bar{c}_{q+2} + (-1)^{d+1}((d+1)!)^2 c_{d-1}^2 \\ &\qquad +(-i)^d 2d!(d+2)!c_{d-2}r = 0, \\ &\sum_{q=0}^{d-2}(2d-2q-1)(q+1)!(2d-q)!c_q\bar{c}_{q+1} + (-i)^d d!(d+1)!c_{d-1}r = 0. \end{aligned} \tag{1.4.16}$$

1.5 Minimal Imbeddings of Spherical Space Forms into Spheres

In this section we apply the equivariant construction to the absolute invariants derived in Section 1.3, in order to obtain isometric minimal *imbeddings* of all three-dimensional homogeneous spherical space forms into spheres. Our exposition follows DeTurck-Ziller [1,2].

Recall from the previous section that the orbit map $f_\xi : S^3 \to S_{W_p}$ defined in (1.4.2) is a minimal immersion of degree p iff the coefficients of ξ in (1.4.1) satisfy the system (1.4.13) of quadratic equations. Let $\xi \in W_p$ be a solution of the system (1.4.13) and assume, in addition, that ξ is an absolute invariant for a finite subgroup G^* of $SU(2)$. The equivariant construction applied to ξ then gives an $SU(2)$-equivariant minimal immersion $f_\xi : S^3 \to S^{2p+1}$ that factors through the action of G^* on S^3, and gives a conformal

minimal immersion $f_\xi : S^3/G^* \to S^{2p+1}$ (denoted by the same symbol). Indeed, using the definition of the orbit map f_ξ and right quaternionic multiplication R, for $g, g' \in S^3 = SU(2)$, we have

$$f_\xi \circ R_g(g') = f_\xi(g'g) = \xi \circ L_{(g'g)^{-1}} = \xi \circ L_{g^{-1}} \circ L_{(g')^{-1}} = f_\xi \circ L_{g^{-1}}(g').$$

Equivalently

$$f_\xi \circ R_g = f_{\xi \circ L_{g^{-1}}}.$$

If $g \in G^*$, then $\xi \circ L_{g^{-1}} = \xi$, so that $f_\xi \circ R_g = f_\xi$. Hence f_ξ factors through the projection $S^3 \to S^3/G^*$, where S^3/G^* is the homogeneous space of right-cosets.

The factored map is an *imbedding* iff G^* is the full invariance group of ξ, i.e. for $g \in SU(2)$, we have $\xi \circ g = \xi$ iff $g \in G^*$.

If ξ lies in R_{2d}, $p = 2d$, then the range dimension reduces to $2d$.

In this section we will compile a comprehensive list of minimal imbeddings of all homogeneous three-dimensional spherical space forms into spheres with minimum range dimension.

The cases of binary tetrahedral, octahedral, and icosahedral groups are technically simpler than the cyclic and binary dihedral groups. In fact, for $G = T$, O, I, choosing ξ the minimum degree absolute invariant in R_{2d}, $d = 3, 4, 6$, the equivariant construction will automatically give a minimal imbedding $f_\xi : S^3/G^* \to S^{2d}$.

For $G = T$, the minimum degree absolute tetrahedral invariant is Ω, given by (1.3.11). We compare a constant multiple of Ω with a typical element in R_6:

$$\xi(z, w) = c_0 z^6 + \bar{c}_0 w^6 + c_1 z^5 w - \bar{c}_1 z w^5 + c_2 z^4 w^2 + \bar{c}_2 z^2 w^4 - i r z^3 w^3, \ r \in \mathbf{R}.$$

Letting $c_0 = c_2 = r = 0$, and c_1 real, we obtain $\xi = c_1 \Omega$. Substituting these coefficients into (1.4.16), we see that the first two equations are equivalent, and they give us

$$c_1 = \frac{1}{4\sqrt{15}}.$$

The last two equations in (1.4.16) are automatically satisfied. We call the orbit map $f_{\Omega/(4\sqrt{15})}$ the *tetrahedral minimal immersion* and denote it by $Tet : S^3 \to S^6$. The full invariance group of Ω is T^* since there are no degree 6 absolute invariants for the larger groups O^*, or I^*. Thus, the tetrahedral minimal immersion defines a minimal imbedding of the tetrahedral manifold S^3/T^* into S^6.

Remark. A result of Moore [1] states that a minimal isometric immersion $f : S^m \to S^n$ with $n \leq 2m - 1$ is totally geodesic (Appendix 2), in particular, the image of f is a great m-sphere in S^n. (Here the metric on S^m is suitably scaled.) The tetrahedral immersion $Tet : S^3 \to S^6$, being a minimum range dimensional (non-totally geodesic) example from S^3, shows

that Moore's lower bound cannot be improved in general. This immersion has an interesting history. Before Moore's lower bound there have been several conjectures as to what the minimum range dimension of a (non-totally geodesic) minimal immersion should be, for example, DoCarmo-Wallach [1] suggested $m(m+3)/2-1$ for immersions of S^m. However, subsequently, Ejiri [1] showed the existence of a minimal immersion of degree 6 into S^6. He also proved that this immersion is *totally real* with respect to the natural almost-complex structure of S^6. His construction was *implicit* as he was using the fundamental theorem for isometric immersions. Mashimo [1] constructed this immersion more explicitly as an $SU(2)$-orbit using the equivariant construction described in this text. He also showed that every totally real immersion of S^3 into S^6 of degree 6 differs from this immersion by an isometry of the range. Dillen-Verstraelen-Vrancken [1] observed that this immersion is a 24-fold cover of S^3 to its image in S^6. The final step was completed by DeTurck-Ziller [1] who, by making use of absolute invariants, proved that the image is the tetrahedral manifold S^3/T^*. Rigidity of the tetrahedral minimal immersion among *all* degree 6 minimal immersions of S^3 into S^6 remains an unsolved problem (see Escher-Weingart [1]).

For $G=O$, the minimum degree absolute octahedral invariant is $\Phi\Psi$ given in (1.3.9). A typical element ξ in R_8 ($d=4$ in (1.4.15)) reduces to a real constant multiple of $\Phi\Psi$ if c_0 is real, $c_4=14c_0$, $c_8=c_0$, and the rest of the coefficients are zero. Substituting $c_0\Phi\Psi$ into (1.4.16), we see that the first two equations are equivalent and give

$$c_0=\frac{1}{96\sqrt{21}},$$

and the last two equations are automatically satisfied. By Theorem 1.2.4, the binary octahedral group O^* is a maximal finite subgroup in $SU(2)$. In particular, we see that O^* is the full invariance group of $\Phi\Psi$. The orbit map gives the *octahedral minimal immersion* $Oct : S^3 \to S^8$, and factoring, we obtain a minimal imbedding of the octahedral manifold S^3/O^* into S^8.

For $G=I$, the minimum degree absolute icosahedral invariant is $\mathcal{I}$ given in (1.3.15). A typical element ξ in R_{12} ($d=6$ in (1.4.15)) reduces to a real constant multiple of $\mathcal{I}$ if c_1 is real, $c_6=11c_1$, $c_{11}=-c_1$, and the rest of the coefficients are zero. Substituting $c_1\mathcal{I}$ into (1.4.16), we see again that the first two equations are equivalent and give

$$c_1=\frac{1}{3600\sqrt{11}},$$

and the last two equations are automatically satisfied. As before, I^* is the full invariance group of $\mathcal{I}$. We arrive at the *icosahedral minimal immersion Ico* $: S^3 \to S^{12}$ that defines a conformal minimal imbedding of the icosahedral manifold S^3/I^* into S^{12}.

Remark 1. It may seem coincidental that, for $G = T, O, I$, the first two equations in (1.4.16) are equivalent and the last two are automatically satisfied. We claim, however, that, for *any* $\xi \in R_{2d}$, $d = 3, 4, 6$, f_ξ is a minimal immersion of degree $2d$, so that the only constraint in (1.4.16) is coming from the first equation (scaling the range of f_ξ to lie in the unit sphere).

Let $G^* \subset S^3 = SU(2)$ be the binary group corresponding to $G = T, O, I$. We first prove that the isotropy representation of G^* on the tangent space $T_o(S^3/G^*)$ at $o = \{G^*\}$ is irreducible. (We will use some basic notions on homogeneous spaces, cf. Helgason [1].) Since G^* is finite, the canonical projection $S^3 \to S^3/G^*$ is a local diffeomorphism, and its differential at $1 \in S^3$ is a linear isomorphism $T_1(S^3) \cong T_o(S^3/G^*)$. This isomorphism carries the isotropy representation of G^* to a linear representation of G^* on $T_1(S^3)$. By the very definition of the isotropy representation, the representation of G^* on $T_1(S^3)$ is given by the differential (at $1 \in S^3$) of conjugations: $q \mapsto gqg^{-1}$, $q \in S^3$, $g \in G^*$. On the other hand, the tangent space $T_1(S^3) \subset \mathbf{H}$ can be identified by the orthogonal complement of $\mathbf{R} \cdot 1$ in $\mathbf{H}$. This is the space $\mathbf{H}_0$ of pure quaternions spanned by $i, j, k \in \subset \mathbf{H}$. The representation of G^* on $\mathbf{H}_0$ is given by conjugation. Summarizing, we see that the isotropy representation of G^* on $T_o(S^3/G^*)$ can be identified with the representation of G^* on $\mathbf{H}_0$ given by quaternionic conjugation. It is well-known that conjugation by a quaternion g with nonzero pure part restricts to a rotation in $\mathbf{H}_0 = \mathbf{R}^3$ (Berger [1], Section 8.9). For $g = a + p$, $a \in \mathbf{R}$, $0 \neq p \in \mathbf{H}_0$, the rotation axis is $\mathbf{R} \cdot p$ and the angle of rotation θ satisfies $\tan(\theta/2) = |p|/|a|$ if $a \neq 0$, and $\theta = \pi$ for $a = 0$. Setting $g = z + jw$, where z and w are given in (1.2.9), the rotation that corresponds to g under the stereographic projection h via Theorem 1.2.1 differs from the rotation defined by conjugation with g by a negative quarter-turn around the second axis. Indeed, on the variables x_0, x_1, x_2, this quarter-turn has the following effect

$$x_0 \mapsto -x_2, \quad x_1 \mapsto x_1, \quad x_2 \mapsto x_0.$$

Performing this in (1.2.9), the quaternion $z + jw$ becomes

$$\left(\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)x_0\right) + j\left(\sin\left(\frac{\theta}{2}\right)(x_1 - ix_2)\right)$$
$$= \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)(ix_0 + jx_1 + kx_2).$$

The real part of this quaternion is $a = \cos(\theta/2)$ and the pure part is $p = \sin(\theta/2)x$. Thus, conjugation by this quaternion has axis $\mathbf{R} \cdot p = \mathbf{R} \cdot x$, $x = (x_0, x_1, x_2)$, and angle of rotation is θ (since $|p|/|a| = \tan(\theta/2)$.).

Summarizing again, we see that the isotropy representation is nothing but the representation of G^* on $\mathbf{R}^3$ given by the binary cover of the group of symmetries of the respective Platonic solid whose position differs form

the one we chose in Section 1.2 by a quarter-turn around the second axis. Irreducibility of the isotropy representation of G^* on S^3/G^* follows. Finally, if $f : S^3 \to S_{R_{2d}}$ is any $SU(2)$-equivariant eigenmap with G^* as an invariance group, where $G = T, O, I$, then f is automatically conformal, thereby a minimal immersion. Indeed, by $SU(2)$-equivariance, conformality needs to be checked only at $T_1(S^3)$. By G^*-invariance, the scalar product on R_{2d} pulled back to $T_o(S^3/G^*)$ by f is G^*-invariant. Since G^* *acts irreducibly on* $T_o(S^3/G^*)$, this pullback must be a constant multiple of the canonical scalar product on $T_o(S^3/G^*) \cong T_1(S^3)$. (This is the real version of Problem 1.30.) This gives us conformality of f.

Remark 2. It is clear that a minimal immersion $f_\xi : S^3 \to S_{R_{2d}}$ of degree $2d$ is a minimum codimension example *among* $SU(2)$*-equivariant minimal immersions* (with harmonic homogeneous polynomial components of degree $2d$). One might be tempted to conjecture that f_ξ is also minimum codimensional among *all* minimal immersions of degree $2d$. Escher-Weingart [1] proved that this was false, providing examples of minimal immersions $f_\xi : S^3 \to S^n$ of degree 36 and $n < 36$.

The cases of cyclic and dihedral groups are more technical. We start with the dihedral group D_d. An easy computation shows that powers of the generators $z^{2d} + w^{2d}$, $zw(z^{2d} - w^{2d})$, and z^2w^2 of the ring of absolute invariants (Section 1.3) do not satisfy (1.4.13) so that we need to take linear combinations. Consider $\xi(z, w) = c_0(z^{2d} + w^{2d}) + c_d z^d w^d$ in W_{2d}. First, assume that d is even. Substituting the coefficients into (1.4.13), we see that the last two equations are automatically satisfied for $d \geq 4$, and the first two give us

$$|c_0|^2 = \frac{d+1}{6d(2d)!} \quad \text{and} \quad |c_d|^2 = \frac{2d-1}{3d(d!)^2}.$$

If we choose both c_0 and c_d real, then ξ lies in $R_{2d} \subset W_{2d}$ since d is even. The equivariant construction applied to ξ thus gives a minimal immersion $f_\xi : S^3 \to S^{2d}$ of degree $2d$ which factors through the orbit map under D_d^*, and defines an *immersion* of the prism manifold S^3/D_d^* into S^{2d}. The question is whether this latter map is an imbedding; or, what is the same, whether D_d^* is the full invariance group of ξ. This is certainly not the case for $d = 4$, since $|c_4|^2/|c_0|^2 = 14^2$, and, in this case, ξ reduces to a constant multiple of $\Phi\Psi$, an absolute invariant of O^*. For $d \geq 6$ even, however, D_d^* is the full invariance group of ξ. Indeed, let G_ξ^* be the full invariance group of ξ. It cannot be cyclic since it contains D_d^*. It cannot be any of T^*, O^*, or I^*, since these do not contain D_d^*, for $d \geq 6$. Finally, if G_ξ^* is dihedral, it must be D_{kd}^*, $k \geq 1$. Since ξ must be an absolute invariant of this group, and, for $k \geq 2$, the only absolute invariant of degree $2d$ for G_ξ^* is $z^d w^d$ (that does not give a minimal immersion), we must have $k = 1$. Summarizing, we obtain that, for $d \geq 6$ even, $f_\xi : S^3 \to S^{2d}$ is a minimal immersion that factors through the orbit map and gives an imbedding of the prism

manifold S^3/D_d^* into S^{2d}. For $d \geq 5$ odd, the invariance group of ξ contains only C_{2d} and not D_d^* by the presence of the odd power of zw. In fact, C_{2d} is the invariance group of ξ as can be shown by an argument similar to the above. Choosing c_0 real and c_d purely imaginary, we land in R_{2d} since d is odd. The equivariant construction applied to ξ gives rise to a minimal imbedding $f_\xi : S^3 \to S^{2d}$ that factors through the orbit map of C_{2d} and defines a minimal imbedding of the lens space $L(2d;1)$ into S^{2d}.

To obtain the missing minimal imbeddings of the prism manifolds S^3/D_d^* for d odd, we set $\xi(z,w) = c_1 zw(z^{2d} - w^{2d}) + c_{d+1} z^{d+1} w^{d+1}$. If $d \geq 7$ then the procedure above works and we arrive at a minimal immersion $f_\xi : S^3 \to S^{2d+2}$ that defines a minimal imbedding of S^3/D_d^* into S^{2d+2}.

The remaining cases of the missing low-order binary dihedral groups and the cyclic groups can be treated similarly. For completeness, we have compiled the final table showing the absolute invariants and the associated minimal imbeddings. Note that the notation is adjusted so that all coefficients are real.

Domain	Absolute Invariant	Coefficients	Range
$L(2d;1)$ d even $d \geq 4$	$c_1 zw(z^{2d}+w^{2d}) + ic_{d+1}z^{d+1}w^{d+1}$	$c_1^2 = \frac{(d+2)(d+4)}{6d^2(d+1)!}$ $c_{d+1}^2 = \frac{(2d+1)(d-2)}{3d^2((d+1)!)^2}$	S^{2d+2}
$L(2d;1)$ d odd $d \geq 5$	$c_0(z^{2d}+w^{2d}) + ic_d z^d w^d$	$c_0^2 = \frac{d+1}{6d(2d)!}$ $c_d^2 = \frac{2d-1}{3d((d)!)^2}$	S^{2d}
$L(d;1)$ d odd	$c_0 z^{3d} + c_d z^{2d} w^d$	$c_0^2 = \frac{d+1}{4d(3d)!}$ $c_d^2 = \frac{3d-1}{4dd!(2d)!}$	S^{6d+1}
$L(4;1)$	$c_0 z^8 + c_6 z^2 w^6$	$c_0 = \frac{1}{72\sqrt{35}}$ $c_6 = \frac{\sqrt{7}}{36\sqrt{10}}$	S^{17}
$L(6;1)$	$c_0 z^6 + c_3 z^3 w^3$	$c_0 = \frac{2}{9\sqrt{5}}$ $c_6 = \frac{2\sqrt{5}}{9}$	S^{13}
S^3/D_d^* d even $d \geq 6$	$c_0(z^{2d}+w^{2d}) + c_d z^d w^d$	$c_0^2 = \frac{d+1}{6d(2d)!}$ $c_d^2 = \frac{2d-1}{3d(d!)^2}$	S^{2d}
S^3/D_d^* d odd $d \geq 7$	$c_1 zw(z^{2d}-w^{2d}) + c_{d+1}z^{d+1}w^{d+1}$	$c_1^2 = \frac{(d+1)(d+2)}{6d^2(2d+1)!}$ $c_{d+1}^2 = \frac{(2d+1)(d-2)}{3d^2((d+1)!)^2}$	S^{2d+2}
S^3/D_2^*	$c_0(14z^4w^4 + 28z^2w^2(z^4+w^4)$ $-3(z^8+w^8))$	$c_0 = \frac{1}{384\sqrt{21}}$	S^8
S^3/D_3^*	$c_1 zw(z^7-w^7) + c_4 z^4 w^4$	$c_1 = \frac{1}{18\sqrt{42}}$ $c_4 = \frac{i\sqrt{7}}{72\sqrt{21}}$	S^8
S^3/D_4^*	$c_0(z^8+w^8+14iz^4w^4)$	$c_0 = \frac{1}{96\sqrt{21}}$	S^{17}
S^3/D_5^*	$c_1(zw(z^{10}-w^{10}) + 11iz^6w^6)$	$c_1 = \frac{1}{3600\sqrt{11}}$	S^{21}
S^3/T^*	$c_0 zw(z^4-w^4)$	$c_0 = \frac{1}{4\sqrt{15}}$	S^6
$S^3/0^*$	$c_0(z^8+14z^4w^4+w^8)$	$c_0 = \frac{1}{96\sqrt{21}}$	S^8
S^3/I^*	$c_1(z^{11}w + 11z^6w^6 - zw^{11})$	$c_1 = \frac{1}{3600\sqrt{11}}$	S^{12}

Scaling the metric on the range appropriately, a conformal immersion becomes an isometric immersion. (Recall that the conformality factor is a constant.) We obtain the following result of DeTurck and Ziller:

Theorem 1.5.1. *Every homogeneous 3-dimensional spherical space form admits a minimal isometric imbedding into a sphere (of appropriate radius).*

Remark. The DeTurck and Ziller's result applies to all domain dimensions. We restricted ourselves to 3-dimensional spherical space forms here for brevity.

1.6 Additional Topic: Klein's Theory of the Icosahedron

According to Galois theory, a polynomial equation is solvable by radicals iff the associated Galois group is solvable. The Galois group of an irreducible quintic (degree 5) polynomial is a subgroup of $\mathcal{S}_5$. Hence, an irreducible quintic polynomial whose roots cannot be obtained by root formulas has Galois group $\mathcal{A}_5$ or $\mathcal{S}_5$. If the Galois group is $\mathcal{S}_5$ then, adjoining the square root of the discriminant to the ground field, the Galois group for this extended ground field becomes $\mathcal{A}_5$.

On the other hand, we saw in Section 1.1 that the symmetry group of the icosahedron is isomorphic with $\mathcal{A}_5$. The question arises naturally whether there is any connection between the the solutions of quintic equations and the icosahedron. This is the subject of Klein's famous Icosahedron Book (Klein [1]). It creates a subtle but explicit correspondence between the two interpretations of the same group $\mathcal{A}_5$: as a Galois group of a nonsolvable irreducible quintic and as the symmetry group of the icosahedron. We devote this section to a sketch of Klein's main result. We will treat the material here somewhat differently than Klein, and rely more on geometry. Unlike the Icosahedron Book, here we take as direct path to the principal result (Klein's "Normalformsatz") as possible. Due to the complexity of the exposition, we divide our treatment into subsections. Other than Klein's wonderful classic, a slightly more detailed account of our approach is in Toth [3]; for a technically different treatment, see Schurman [1]. We will use some basic concepts and facts from Galois theory; the principal reference for this is Artin [1].

A. Polyhedral Equations

In Section 1.3, for each finite Möbius group G we constructed a rational function $q = q_G : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ with invariance group G. Geometrically, q is the projection of a holomorphic $|G|$-fold branched covering between Riemann spheres, and the branch values are $Z = 0$, $Z = 1$, and $Z = \infty$, with branch

numbers ν_2, ν_1, and ν_0 minus 1. (In the case of the dihedron ν_0 and ν_2 are switched.) As noted in the proof of Theorem 1.3.1, for a given $Z \in \mathbf{C}$, the equation $q(\zeta) = Z$ for ζ can be written as a degree $|G|$ polynomial equation

$$P(\zeta) - ZQ(\zeta) = 0, \tag{1.6.1}$$

where $q = P/Q$ with P and Q being the polynomial numerator and denominator of q (with no common factors). We call (1.6.1) the *polyhedral equation associated to* G. Clearly, this equation has $|G|$ solutions counted with multiplicity, and depending on the parameter Z. We now discuss solvability of (1.6.1) in each particular case of G.

The case of the cyclic group C_d is obvious, since the associated equation is $\zeta^d = Z$, and the solutions are simply the d-th roots of Z.

We now consider the equation of the dihedron. Using the second table in Section 1.3 along with (1.3.2) and (1.3.4), $q = q_{D_d}$ works out as

$$\begin{aligned} q_{D_d}(\zeta) &= -\frac{\alpha(z,w)^2}{\gamma(z,w)^d} = -\frac{(z^d - w^d)^2}{4z^d w^d} \\ &= -\frac{(\zeta^d - 1)^2}{4\zeta^d} = Z, \end{aligned} \tag{1.6.2}$$

where, as usual, $\zeta = z/w$, $z, w \in \mathbf{C}$. Multiplying out, we obtain the equation of the dihedron

$$\zeta^{2d} + 2(2Z - 1)\zeta^d + 1 = 0,$$

which is a quadratic equation in ζ^d. The solutions are

$$\zeta = q_{D_d}^{-1}(Z) = \sqrt[d]{1 - 2Z \pm 2\sqrt{Z(Z+1)}}.$$

We see that inverting q_{D_d} amounts to extracting a square root followed by the extraction of a d-th root.

For the equation of the tetrahedron, using (1.3.7)-(1.3.8), we have

$$\begin{aligned} q_T(\zeta) &= \frac{\Psi(z,w)^3}{\Phi(z,w)^3} \\ &= \left(\frac{z^4 + 2\sqrt{3}iz^2w^2 + w^4}{z^4 - 2\sqrt{3}iz^2w^2 + w^4}\right)^3 \\ &= \left(\frac{\zeta^4 + 2\sqrt{3}i\zeta^2 + 1}{\zeta^4 - 2\sqrt{3}i\zeta^2 + 1}\right)^3 = Z. \end{aligned}$$

Taking cubic roots of both sides, we obtain a quadratic equation in ζ^2 that can be easily solved. Inverting q_T amounts to extracting a cubic root followed by the extraction of two square roots. Comparing the expression of q_T just obtained with (1.6.2) for $d = 2$, we have

$$q_T(\zeta) = \left(\frac{q_{D_2}(\zeta) - e^{i\frac{2\pi}{3}}}{q_{D_2}(\zeta) + e^{i\frac{2\pi}{3}}}\right)^3.$$

As noted in Section 1.3, the octahedral invariants can be written as polynomials in the tetrahedral invariants. Using the second table in Section 1.3 again, we have

$$\frac{q_O(\zeta)}{q_O(\zeta)-1} = \frac{(\Phi\Psi)^3}{((\Phi^3+\Psi^3)/2)^2} = \frac{(\Psi/\Phi)^3}{((\Psi/\Phi)^3+1)/2)^2} = \frac{Z}{Z-1},$$

where we omitted the arguments z, w for simplicity. Multiplying out, we obtain a quadratic equation in $(\Psi/\Phi)^3$. Since this is $q_T(\zeta)$, we see that inverting q_O amounts to extracting a square root, followed by the extraction of a cubic root, and two square roots.

Remark. In view of Galois theory, the specific sequence of roots is not surprising since O has a composition series $O \cong \mathcal{S}_4 \supset \mathcal{A}_4 \supset D_2 \supset C_2 \supset \{e\}$, and the indices of the consecutive normal subgroups are $2, 3, 2, 2$.

For the icosahedron, we have

$$q_I(\zeta) = \frac{\mathcal{H}^3}{1728\,\mathcal{I}^5} = Z.$$

Multiplying out, we obtain the *equation of the icosahedron*

$$\mathcal{H}^3(\zeta,1) - 1728\,Z\,\mathcal{I}^5(\zeta,1) = 0.$$

Using the explicit forms (1.3.15)-(1.3.17) of the icosahedral invariants, this rewrites as

$$\begin{aligned}((\zeta^{20}+1) - 228(\zeta^{15}-\zeta^5) + 494\zeta^{10})^3 \\ +1728Z\zeta^5(\zeta^{10}+11\zeta-1)^5 = 0.\end{aligned} \tag{1.6.3}$$

Due to the presence of Z, a natural ground field for this equation is the field of rational functions $k(Z)$ over $k = \mathbf{Q}(\omega)$, where $\omega = e^{i\frac{2\pi}{5}}$ is a fifth root of unity. Here we adjoined ω to $\mathbf{Q}$ to ensure that the linear fractional transformations (1.2.23) in the icosahedral group $I = \mathcal{A}_5$ are *defined over* k (i.e., the coefficients in (1.2.23) are in k). Since $q_I(\zeta) = Z$, and q_I is rational over k, the field of rational functions $k(\zeta)$ is an extension field of $k(Z)$. The icosahedral group I acts on $k(\zeta)$ by $k(Z)$-automorphisms (automorphisms of the field $k(\zeta)$ that leave $k(Z)$ pointwise fixed), where $g \in I$ acts on $r \in k(\zeta)$ as $g \cdot r = r \circ g^{-1}$. In fact, since invariance group of q_I is I, the fixed field is $k(\zeta)^I = k(Z)$. We obtain that $k(\zeta)/k(Z)$ is a *Galois extension with Galois group* I. The equation of the icosahedron has *splitting field* $k(\zeta)$ over $k(Z)$; the solutions are linear fractions that represent the linear fractional transformations in I. Since I is transitive on the solutions, we see that the equation of the icosahedron is *irreducible over* $k(Z)$. (Notice that this argument applies to the other polyhedral equations as well. Hence all polyhedral equations are irreducible over $k(Z)$, where k is the appropriate ground field.) By Galois theory, the equation of the

icosahedron *cannot be solved by radicals* since the Galois group $I \cong \mathcal{A}_5$ is not solvable. (In contrast, as we have seen above, the equations of the dihedron, tetrahedron, and octahedron are solvable by radicals, since the octahedral group is solvable.)

The question arises naturally as to what kind of additional "transcendental" procedure is needed to express the solutions in an explicit form. In this subsection we will briefly indicate that any solution of a polyhedral equation can be written as the quotient of two linearly independent solutions of a homogeneous second-order linear differential equation with exactly three singular points, all regular. These differential equations are called *hypergeometric* (Ahlfors [1]). Equivalently, we will show that, for each spherical Platonic tessellation, the inverse q^{-1} of the rational function q is the quotient of two hypergeometric functions.

Consider the *Schwarzian* $\mathcal{S}(f)$ of a (possibly multiple valued) holomorphic function f, defined by

$$\mathcal{S}(f) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

It is well-known that $\mathcal{S}(f)$ is invariant under *any* linear fractional transformation, in fact, this property can be used to define $\mathcal{S}$ (Klein [1]). Let G be a finite Möbius group, and consider the holomorphic branched covering $q = q_G : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ with invariance group G. By definition, for given Z, the solutions of the associated polyhedral equation are simply the elements in the inverse image $q^{-1}(Z)$, a single G-orbit. By G-invariance, when we compose q^{-1} with a linear fractional transformation in G, we pass from a single valued branch of q^{-1} to another. By the characteristic property of the Schwarzian, the function $s = \mathcal{S}(q^{-1})$ must be single-valued. Since it is a self-map of $\hat{\mathbf{C}}$ its restriction to $\mathbf{C}$ must be a rational function. The poles of s (at $Z = 0$, $Z = 1$, and $Z = \infty$) can be calculated explicitly by expanding q^{-1} into Laurent series around these points, and differentiating according to the recipe provided by the Schwarzian. Following Riemann, the poles determine s uniquely so that an explicit form of s can be derived. It is also well-known that the solutions of the third-order differential equation

$$\mathcal{S}(q^{-1}) = s$$

are linear fractional transformations applied to linearly independent solutions of a homogeneous second-order linear differential equation

$$z'' = a(Z)z' + b(Z)z, \tag{1.6.4}$$

where a and b are rational functions with

$$s = a' - \frac{1}{2}a^2 - 2b \tag{1.6.5}$$

(again, see Ahlfors [1]). Setting $a(Z) = 1/Z$, and b satisfying (1.6.5), it turns out that (1.6.4) has exactly 3 singular points, all regular. Thus, this

particular differential equation is *hypergeometric*, and the solutions are hypergeometric functions. We obtain that q^{-1} can be written as a quotient of hypergeometric functions.

Remark. In 1873 Schwarz classified all hypergeometric differential equations with finite monodromy groups. The reflection principle (named after him) applied to holomorphic solutions of a hypergeometric differential equation (with singularities at 0, 1, ∞) on the upper half-plane gives Klein's fundamental rational function q.

B. The Tschirnhaus Transformation

We are interested in reducing the general irreducible quintic

$$\zeta^5 + a_1\zeta^4 + a_2\zeta^3 + a_3\zeta^2 + a_4\zeta + a_5 = 0, \tag{1.6.6}$$

to a simpler form

$$\zeta^5 + \tilde{a}_1\zeta^4 + \tilde{a}_2\zeta^3 + \tilde{a}_3\zeta^2 + \tilde{a}_4\zeta + \tilde{a}_5 = 0, \tag{1.6.7}$$

in which some (but not all) coefficients vanish. This reduction will be accomplished by the so-called *Tschirnhaus transformation*. The general Tschirnhaus transformation is given by a polynomial $\tilde{\zeta}$ in ζ of degree ≤ 4 with coefficients in $\mathbf{Q}[a_1, \ldots, a_5]$. The coefficients depend on parameters $\lambda_1, \ldots, \lambda_4$ to be chosen appropriately. We define

$$\tilde{\zeta} = \sum_{l=1}^{4} \lambda_l \zeta^{(l)}, \tag{1.6.8}$$

where

$$\zeta^{(l)} = \zeta^l - \frac{1}{5}\sum_{j=0}^{4} \zeta_j^l, \quad l = 1, \ldots, 4, \tag{1.6.9}$$

and $\zeta_0, \ldots, \zeta_4$ are the roots of (1.6.6). $\tilde{a}_1 = 0$, $\tilde{a}_2 = 0$, etc. in (1.6.7) amount to polynomial equations in the coefficients $\lambda_1, \ldots, \lambda_4$. The polynomials involved will be of degree ≤ 4, so that the equations can be solved by root formulas.

We begin by noticing that on the right-hand side in (1.6.9) the sum of powers is a symmetric polynomial in the roots, and thus, by the fundamental theorem on symmetric polynomials, it can be expressed as a polynomial in the coefficients $a_1, \ldots, a_5$ in (1.6.6). For example, since

$$\sum_{l=0}^{4} \zeta_l = -a_1, \quad \sum_{l=0}^{4} \zeta_l^2 = a_1^2 - 2a_2,$$

we have

$$\zeta^{(1)} = \zeta + \frac{a_1}{5}, \quad \zeta^{(2)} = \zeta^2 - \frac{1}{5}\left(a_1^2 - 2a_2\right). \tag{1.6.10}$$

The Tschirnhaus transformation $\tilde{\zeta}$ acts on (1.6.6) by transforming its roots ζ_j, to the roots $\tilde{\zeta}_j$ of (1.6.7), where

$$\tilde{\zeta}_j = \sum_{l=1}^{4} \lambda_l \zeta_j^{(l)} = \sum_{l=1}^{4} \lambda_l \left(\zeta_j^l - \frac{1}{5} \sum_{k=0}^{4} \zeta_k^l \right), \quad j = 0, \ldots, 4. \tag{1.6.11}$$

Since

$$\sum_{j=0}^{4} \tilde{\zeta}_j = \sum_{l=1}^{4} \lambda_l \sum_{j=0}^{4} \left(\zeta_j^l - \frac{1}{5} \sum_{k=0}^{4} \zeta_k^l \right) = 0,$$

we have

$$\tilde{a}_1 = 0$$

for *any* Tschirnhaus transformation.

Setting $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = \lambda_4 = 0$ in (1.6.8), the simplest Tschirnhaus transformation is

$$\tilde{\zeta} = \zeta^{(1)} = \zeta + \frac{a_1}{5}.$$

This makes only $\tilde{a}_1$ vanish.

Next, we look for a Tschirnhaus transformation which gives $\tilde{a}_2 = 0$. Setting $\lambda_1 = \lambda$, $\lambda_2 = 1$, $\lambda_3 = \lambda_4 = 0$, our Tschirnhaus transformation takes the form

$$\begin{aligned} \tilde{\zeta} &= \lambda \zeta^{(1)} + \zeta^{(2)}, \\ &= \lambda \left(\zeta + \frac{a_1}{5} \right) + \zeta^2 - \frac{1}{5} \left(a_1^2 - 2a_2 \right), \end{aligned} \tag{1.6.12}$$

where $\lambda \in \mathbf{C}$ is a parameter to be determined. Since $\tilde{a}_1 = 0$, the vanishing of $\tilde{a}_2$ in (1.6.7) amounts to the vanishing of $\sum_{j=0}^{4} \tilde{\zeta}_j^2$. This gives the following quadratic equation for λ:

$$\begin{aligned} \sum_{j=0}^{4} \tilde{\zeta}_j^2 &= \sum_{j=0}^{4} (\lambda \zeta_j^{(1)} + \zeta_j^{(2)})^2 \\ &= \lambda^2 \sum_{j=0}^{4} \left(\zeta_j^{(1)} \right)^2 + 2\lambda \sum_{j=0}^{4} \zeta_j^{(1)} \zeta_j^{(2)} \\ &\quad + \sum_{j=0}^{4} \left(\zeta_j^{(2)} \right)^2 = 0. \end{aligned}$$

Again, by the fundamental theorem on symmetric polynomials, the coefficients of the quadratic polynomial in λ depend only on $a_1, \ldots, a_5$. The corresponding quadratic equation can be solved for λ in terms of $a_1, \ldots, a_5$.

The two solutions for λ involve the square root of the quadratic expression

$$4\left(\sum_{j=0}^{4}\zeta_j^{(1)}\zeta_j^{(2)}\right)^2 - 4\sum_{j=0}^{4}(\zeta_j^{(1)})^2\sum_{j=0}^{4}(\zeta_j^{(2)})^2.$$

This is the discriminant δ multiplied by $(\sum_{j=0}^{4}(\zeta_j^{(1)})^2)^2$. We have $\delta \in k$ (since $k \supset \mathbf{Q}[a_1, \ldots, a_5]$) but, in general, $\sqrt{\delta} \notin k$ so that $\sqrt{\delta}$ *needs to be adjoined to the ground field* k.

Remark 1. Bring (in 1786) and Jerrard (in 1834) showed independently that a suitable Tschirnhaus transformation can make $\tilde{a}_1$, $\tilde{a}_2$, and $\tilde{a}_3$ simultaneously vanish. The corresponding quintic

$$\zeta^5 + \tilde{a}_4\zeta + \tilde{a}_5 = 0,$$

is called the *Bring-Jerrard form*. By scaling, the Bring-Jerrard form can be further reduced to the special quintic

$$z^5 + z - c = 0.$$

A root of this polynomial is called an *ultraradical* and it is denoted by $\sqrt[*]{c}$. Using this equation, an ultraradical can be easily expanded into a convergent series. Bring and Jerrard thus showed that the general quintic can be solved by radicals and ultraradicals. The relation of this special quintic to the so-called modular equation was used by Hermite who pointed out that the general quintic can be solved in terms of elliptic modular functions. In our discussion we restrict ourselves to the two simplest Tschirnhaus transformations.

Remark 2. Tschirnhaus transformations can also be used to obtain a reduced quintic with $\tilde{a}_1 = \tilde{a}_3$. This is called a Brioschi quintic. The approach that uses this reduction of the general irreducible quintic is given in Fricke [1], Vol. II, and Schurman [1].

Summarizing (and adjusting the notation) the problem of solvability of the general quintic can be reduced (at the expense of a quadratic extension of the ground field) to the solvability of the equation

$$P(\zeta) = \zeta^5 + 5a\zeta^2 + 5b\zeta + c = 0, \tag{1.6.13}$$

where we inserted the numerical factors for later convenience. A quintic such as this with vanishing degree 3, and four terms is said to be *canonical*. For future reference we include here the discriminant

$$\delta = \prod_{0\le j<l\le 4}(\zeta_j - \zeta_l)^2 \tag{1.6.14}$$

of (1.6.13) as a polynomial in the coefficients a, b, c:

$$\frac{\delta}{5^5} = 108a^5c - 135a^4b^2 + 90a^2bc^2 - 320ab^3c + 256b^5 + c^4. \tag{1.6.15}$$

This formula can be obtained by a somewhat tedious but elementary computation.

Consider the roots $\zeta_0, \ldots, \zeta_4$ of the canonical equation (1.6.13) as homogeneous coordinates of a point $[\zeta_0 : \ldots : \zeta_4]$ in the complex projective 4-space $\mathbf{C}P^4$. (In what follows we will use some basic facts in projective geometry; cf. Berger [1], Section 14.1.) The trivial case $P(\zeta) = \zeta^5$ needs to be excluded since it gives $\zeta_0 = \ldots = \zeta_4 = 0$. Since ordering the roots cannot be prescribed universally, there are 120 projective points obtained from one another by permuting the coordinates which correspond to the solutions of (1.6.13). To incorporate this ambiguity, we consider the symmetric group $\mathcal{S}_5$ acting on $\mathbf{C}P^4$ by permuting the homogeneous coordinates. Thus the 120 points form an orbit in $\mathbf{C}P^4$ under the action of $\mathcal{S}_5$.

Since our equation is canonical, the roots $\zeta_0, \ldots, \zeta_4$ satisfy the relations

$$\sum_{j=0}^{4} \zeta_j = 0, \quad \sum_{j=0}^{4} \zeta_j^2 = 0. \tag{1.6.16}$$

Hence $[\zeta_0 : \ldots : \zeta_4]$ lies in the (smooth) complex surface

$$\mathcal{Q}_0 = \left\{ [\zeta_0 : \ldots : \zeta_4] \in \mathbf{C}P^4 \;\middle|\; \sum_{j=0}^{4} \zeta_j = \sum_{j=0}^{4} \zeta_j^2 = 0 \right\}.$$

$\mathcal{Q}_0$ can be identified with the *standard complex projective quadric* once we identify the complex projective 3-space $\mathbf{C}P^3$ with the linear slice $\mathbf{C}P_0^3$ of $\mathbf{C}P^4$ defined by $\sum_{j=0}^{4} \zeta_j = 0$. We thus set

$$\mathbf{C}P_0^3 = \left\{ [\zeta_0 : \ldots : \zeta_4] \in \mathbf{C}P^4 \;\middle|\; \sum_{j=0}^{4} \zeta_j = 0 \right\}. \tag{1.6.17}$$

With this we can interpret the Tschirnhaus transformation. We start with the point $[\zeta_0 : \ldots : \zeta_4] \in \mathbf{C}P^4$ whose homogeneous coordinates are the roots of a general quintic. Using (1.6.9), we form the projective points $[\zeta_0^{(1)} : \ldots : \zeta_4^{(1)}]$ and $[\zeta_0^{(2)} : \ldots : \zeta_4^{(2)}]$ that both lie in $\mathbf{C}P_0^3$. The projective line passing through these two points intersects the quadric $\mathcal{Q}_0$ in two points. The Tschirnhaus transformation associates to $[\zeta_0 : \ldots : \zeta_4]$ one of the intersection points. As emphasized above, this process generally requires a quadratic extension of the ground field k. The entire construction is invariant under the action of $\mathcal{S}_5$. This follows because, under the action of $\mathcal{S}_5$, the coordinates of the points $[\zeta_0^{(l)} : \ldots : \zeta_4^{(l)}]$, $l = 0, \ldots, 4$, are permuted the same way as the coordinates of $[\zeta_0 : \ldots : \zeta_4]$.

C. Quintic Resolvents of the Icosahedral Equation

If K/k is a Galois extension with Galois group G, then for $t_0 \in K - k$, we can consider the orbit

$$G(t_0) = \{t_0, \ldots, t_{n_0}\},$$

and the polynomial

$$P^*(X) = \prod_{j=0}^{n_0} (X - t_j).$$

Then $P^* \in k[X]$ is irreducible and $[k(t_0) : k] = n_0 + 1$. In particular, $n_0 + 1$ divides $|G|$. If K is the splitting field of a polynomial $P \in k[\zeta]$ with roots $\zeta_0, \ldots, \zeta_n$, then t_0 *is a rational function of* $\zeta_0, \ldots, \zeta_n$. In this case we call P^* a *resolvent polynomial* of P (or the associated polynomial equation $P = 0$).

Since the computations in this subsection are going to be involved, we will sketch out our general plan. Recall that our main objective is to establish a connection between the (irreducible) quintic in canonical form

$$P(\zeta) = \zeta^5 + 5a\zeta^2 + 5b\zeta + c$$

and the solutions of the icosahedral equation

$$\mathcal{H}^3(\zeta, 1) - 1728 Z \mathcal{I}^5(\zeta, 1) = 0.$$

Since the latter is a polynomial equation of degree 60, we must find a suitable *quintic* resolvent of $\mathcal{H}^3 - 1728 Z \mathcal{I}^5$ *directly comparable to our canonical quintic.* Our task is actually harder since P depends on the three complex coefficients a, b, c. Thus, we need to find a quintic resolvent in canonical form that contains, in addition to Z, two extra parameters λ and μ, say. We write this resolvent as

$$P^*(X) = X^5 + 5a(Z, \lambda, \mu)X^2 + 5b(Z, \lambda, \mu)X + c(Z, \lambda, \mu).$$

We expect the *coefficients* to be explicitly computable rational functions in Z, λ, μ. Since P^* is an icosahedral resolvent, we also need to be able to express the *zeros* of P^* in terms of the solutions of the icosahedral equation; or, what is the same, in terms of hypergeometric functions.

Once P^* is worked out explicitly, the matching with the irreducible quintic P above amounts to solving the system

$$\begin{aligned} a &= a(Z, \lambda, \mu), \\ b &= b(Z, \lambda, \mu), \\ c &= c(Z, \lambda, \mu). \end{aligned}$$

This we will be able to carry out with the additional constraint

$$\sqrt{\delta} = \sqrt{\delta(Z, \lambda, \mu)},$$

where $\delta(Z, \lambda, \mu)$ is the discriminant of P^*. This matching process will establish that the roots of P and P^* coincide *as sets.* To obtain a root-by-root

match we will also need to look at how the Galois group $\mathcal{A}_5$ acts on these two sets of roots.

The five roots of a quintic resolvent constitute an $\mathcal{A}_5$-orbit in the extension $k(\zeta)/k(Z)$. We thus need to find five rational functions in ζ that are permuted amongst themselves by the icosahedral substitutions. During the construction of the rational function q, we learned that it is much easier to construct forms first. Our task thus reduces to finding an $\mathcal{A}_5$-orbit of five forms. This will give us a "quintic resolvent." A further advantage to using forms is that we can derive them from geometric situations.

Recall from Section 1.1 the five configurations of three mutually perpendicular golden rectangles inscribed in the icosahedron. Each configuration defines an octahedron. The vertices of the octahedron are the midpoints of the shorter edges of the golden rectangles. These edges are also edges of the icosahedron. For each octahedron (projected radially to S^2 and then to $\hat{\mathbf{C}}$) we can construct an octahedral form of degree 6 that vanishes on the 6 vertices. The five octahedral forms (corresponding to the five octahedra) will be the roots of our quintic resolvent.

Recall the half-turns U, V, W introduced in Section 1.2. The linear fractional transformation corresponding to U is $\zeta \mapsto -1/\zeta$, while those that correspond to V and W are given in (1.2.21) and (1.2.22). Since these half-turns are (commuting) symmetries of the icosahedron their (mutually perpendicular) axes pass through the vertices of one of the five octahedra. We choose this as our first octahedron (projected to $\hat{\mathbf{C}}$) so that its vertices are the fixed points of these half-turns. The fixed points of U are the solutions of $\zeta^2 = -1$, and a form that vanishes on these points is $z^2 + w^2$, $z, w \in \mathbf{C}$, $\zeta = z/w$. To obtain the form that vanishes on the fixed points of V, homogenizing (1.2.21), we need to solve

$$\frac{(\omega^2 - \omega^3)z + (\omega - \omega^4)w}{(\omega - \omega^4)z - (\omega^2 - \omega^3)w} = \frac{(\omega + \omega^4)z + w}{z - (\omega + \omega^4)w} = \frac{z}{w}.$$

Multiplying out, we have

$$z^2 - 2(\omega + \omega^4)zw - w^2 = 0.$$

Using (1.2.17), we obtain that the fixed points of V are given by the zeros of the quadratic form

$$z^2 - \frac{2}{\tau}zw - w^2,$$

where τ is the golden section.

In a similar vein, the fixed points of W are given by the zeros of the quadratic form

$$z^2 + 2\tau zw - w^2.$$

Putting all these together, we define the first octahedral form Ω_0 as the product of these 3 quadratic forms:

$$\begin{aligned}\Omega_0(z,w) &= (z^2+w^2)\left(z^2-\frac{2}{\tau}zw-w^2\right)\left(z^2+2\tau zw-w^2\right)\\ &= (z^2+w^2)(z^4+2z^3w-6z^2w^2-2zw^3+w^4),\end{aligned} \tag{1.6.18}$$

where we used the defining equality $\tau - 1/\tau = 1$ of the golden section.

The remaining four octahedral forms Ω_j, $j = 1,\dots,4$, are obtained by applying to Ω_0 the homogeneous substitutions

$$z \mapsto \pm\omega^{3j}z, \quad w \mapsto \pm\omega^{2j}w$$

corresponding to the rotations $S^j : \zeta \mapsto \omega^j\zeta$, $j = 1,\dots,4$, in (1.2.15). (Recall that S^j corresponds to the two diagonal matrices in $SU(2)$ with diagonal elements $\pm\omega^{3j}$, $\pm\omega^{2j}$.) After some computations we arrive at the five octahedral forms

$$\begin{aligned}\Omega_j(z,w) = {}& \omega^{3j}z^6+2\omega^{2j}z^5w-5\omega^jz^4w^2\\ & -5\omega^{4j}z^2w^4-2\omega^{3j}zw^5+\omega^{2j}w^6,\ j=0,\dots,4.\end{aligned} \tag{1.6.19}$$

By definition, the substitution corresponding to S permutes Ω_j, $j = 0,\dots,4$, cyclically. (Geometrically, the midpoints of the five edges that meet at the north pole N are vertices of the five octahedra.) It is also clear that $U, V, W(= UV)$ all fix Ω_0, since the zero set of Ω_0 consists of the fixed points of U, V, W. As shown by computation, W switches Ω_1 and Ω_2, and also Ω_3 and Ω_4 (Problem 1.32). We express these as follows

$$\begin{aligned}&S:\ \Omega_j \mapsto \Omega_{j+1(\bmod 5)},\ j=0,\dots,4,\\ &W:\ \Omega_0\mapsto\Omega_0,\ \Omega_1\leftrightarrow\Omega_2,\ \Omega_3\leftrightarrow\Omega_4.\end{aligned} \tag{1.6.20}$$

Since S and W generate the icosahedral group, it is clear that $\{\Omega_j\}_{j=0}^4$ constitutes a single $\mathcal{A}_5$-orbit. Following the general recipe of constructing the resolvent, we now need to consider the quintic polynomial with "roots" Ω_j, $j = 0,\dots,4$. This resolvent quintic will not be canonical, however, since $\sum_{j=0}^4 \Omega_j^2 \neq 0$. (Nevertheless, this resolvent is of great interest since $\sum_{j=0}^4 \Omega_j = 0$ and $\sum_{j=0}^4 \Omega_j^3 = 0$; see Problem 1.34.)

As noted above, our task is to construct a quintic resolvent in *canonical form*. To construct a canonical quintic resolvent from the octahedral forms is remarkably simple. In fact, the Hessians $\Xi_j = \mathit{Hess}\,(\Omega_j)$, $j = 0,\dots,4$, are of degree 8, and they also constitute a single $\mathcal{A}_5$-orbit. They clearly satisfy the analogue of (1.6.16):

$$\sum_{j=0}^4 \Xi_j = 0,\ \sum_{j=0}^4 \Xi_j^2 = 0, \tag{1.6.21}$$

since, by (1.3.19), there are no icosahedral forms in degrees 8 and 16. We include here the explicit form

$$\begin{aligned}\Xi_j(z,w) = &-\omega^{4j}z^8 + \omega^{3j}z^7w - 7\omega^{2j}z^6w^2 - 7\omega^j z^5w^3 \\ &+7\omega^{4j}z^3w^5 - 7\omega^{3j}z^2w^6 - \omega^{2j}zw^7 - \omega^j w^8,\ j = 0,\dots,4.\end{aligned} \tag{1.6.22}$$

This is obtained by a tedious computation. (Problem 1.33 asks for the geometric interpretation of the zeros of Ξ_j, $j = 0,\dots,4$.)

To get a step further, we also notice that

$$\sum_{j=0}^{4}\Omega_j\Xi_j = 0,\ \sum_{j=0}^{4}\Omega_j\Xi_j^2 = 0,\ \sum_{j=0}^{4}(\Omega_j\Xi_j)^2 = 0, \tag{1.6.23}$$

since left-hand sides are icosahedral invariants of degrees 14, 22 and 28, respectively, and, by (1.3.19) again, there are no icosahedral invariants in these degrees. For greater generality, we thus seek a resolvent for the linear combinations

$$\Upsilon_j = u\,\Xi_j + v\,\Omega_j\Xi_j,\ \ j = 0,\dots,4. \tag{1.6.24}$$

where, by homogeneity, the coefficients u and v are forms of degrees 30 and 24. In fact, we will put

$$u = 12\lambda\mathcal{J},\ \ v = 144\mu\mathcal{I}^2, \tag{1.6.25}$$

where λ and μ are complex parameters, but, for the time being, we work with (1.6.24). We write the resolvent polynomial as

$$\prod_{j=0}^{4}(\mathcal{X} - \Upsilon_j) = \mathcal{X}^5 + b_1\mathcal{X}^4 + b_2\mathcal{X}^3 + b_3\mathcal{X}^2 + b_4\mathcal{X} + b_5,$$

where we used $\mathcal{X}$ as variable. By what we said above, the relations (1.6.21) and (1.6.23) imply that $b_1 = b_2 = 0$. Expanding, and using the fact that the coefficients must be invariants of the icosahedral group, after somewhat tedious computations, we arrive at the following

$$\begin{aligned}&\mathcal{X}^5 + 5\mathcal{X}^2(8u^3\mathcal{I}^2 + u^2v\mathcal{J} + 72uv^2\mathcal{I}^3 + v^3\mathcal{I}\mathcal{J}) \\ &\quad + 5\mathcal{X}(-u^4\mathcal{I}\mathcal{H} + 18u^2v^2\mathcal{I}^2\mathcal{H} + uv^3\mathcal{H}\mathcal{J} + 27v^4\mathcal{I}^3\mathcal{H}) \\ &\quad + (u^5\mathcal{H}^2 - 10u^3v^2\mathcal{I}\mathcal{H}^2 + 45uv^4\mathcal{I}^2\mathcal{H}^2 + v^5\mathcal{J}\mathcal{H}^2).\end{aligned}$$

We introduce the new variable

$$X = \frac{\mathcal{I}}{\mathcal{J}\mathcal{H}}\mathcal{X}$$

along with

$$r_j = 12\frac{\mathcal{I}^2}{\mathcal{J}}\Omega_j,\quad s_j = 12\frac{\mathcal{I}}{\mathcal{H}}\Xi_j,\ \ j = 0,\dots,4. \tag{1.6.26}$$

Combining these with (1.6.24) and (1.6.25), we obtain

$$\Upsilon_j = \frac{\mathcal{J}\mathcal{H}}{\mathcal{I}} t_j, \quad j = 0, \ldots, 4,$$

where

$$t_j = \lambda s_j + \mu r_j s_j, \quad j = 0, \ldots, 4. \tag{1.6.27}$$

In terms of these new variables, our resolvent polynomial becomes

$$\begin{aligned} P^*(X) &= \prod_{j=0}^{4} (X - t_j) \qquad (1.6.28) \\ &= X^5 + \frac{5X^2}{Z}\left(8\lambda^3 + 12\lambda^2\mu + \frac{6\lambda\mu^2 + \mu^3}{1 - Z}\right) \\ &+ \frac{15X}{Z}\left(-4\lambda^4 + \frac{6\lambda^2\mu^2 + 4\lambda\mu^3}{1 - Z} + \frac{3\mu^4}{4(1 - Z)^2}\right) \\ &+ \frac{3}{Z}\left(48\lambda^5 - \frac{40\lambda^3\mu^2}{1 - Z} + \frac{15\lambda\mu^4 + 4\mu^5}{(1 - Z)^2}\right). \end{aligned}$$

This is called the *canonical resolvent polynomial* of the icosahedral equation (1.6.3). By homogeneity of (1.6.26), *the roots* $t_j = t_j(\zeta, \lambda, \mu)$, $j = 0, \ldots, 4$, of the resolvent P^* *are rational functions in* $\zeta = z/w$, and *linear in* λ *and* μ.

As in the case of the octahedral forms, the icosahedral generators S and W induce the following permutations on the roots:

$$\begin{aligned} S &: \ t_j \mapsto t_{j+1 (\bmod 5)}, \ j = 0, \ldots, 4, \qquad (1.6.29) \\ W &: \ t_0 \mapsto t_0, \ t_1 \leftrightarrow t_2, \ t_3 \leftrightarrow t_4. \end{aligned}$$

Continuing the analogy with the canonical quintic, we can describe the solution set t_j, $j = 0, \ldots, 4$, using homogeneous coordinates $[t_0 : \ldots : t_4]$ in $\mathbf{C}P^4$. The projective points corresponding to the solutions lie in the complex projective quadric

$$\mathcal{Q}_0 = \left\{ [t_0 : \ldots : t_4] \in \mathbf{C}P^4 \;\middle|\; \sum_{j=0}^{4} t_j = \sum_{j=0}^{4} t_j^2 = 0 \right\}. \tag{1.6.30}$$

Finally, since the solution set contains two (linear) parameters, it is clear that the projective points corresponding to the zeros of the canonical resolvent fill $\mathcal{Q}_0$.

In summary, we found that the complex projective quadric $\mathcal{Q}_0$ in $\mathbf{C}P_0^3 (\subset \mathbf{C}P^4)$ serves as a "parameter space" for the solutions of the canonical resolvent of the icosahedral equation. Since the solutions depend rationally on the solutions of the icosahedral equation, our remaining task is to find a matching parametrization of the solutions of the quintic in canonical form with $\mathcal{Q}_0$.

We write (1.6.28) in the short form:

$$P^*(X) = \prod_{j=0}^{4}(X - t_j) = X^5 + 5a(Z,\lambda,\mu)X^2 + 5b(Z,\lambda,\mu)X + c(Z,\lambda,\mu).$$

Let $\delta(Z,\lambda,\mu)$ be the discriminant of the canonical quintic resolvent. It has the same expression as δ in (1.6.15) with a, b, c replaced by $a(Z,\lambda,\mu), b(Z,\lambda,\mu), c(Z,\lambda,\mu)$.

To match the zeros of P^* in (1.6.28) and the zeros of the canonical quintic in (1.6.13) we equate the coefficients

$$\begin{aligned} a &= a(Z,\lambda,\mu), \\ b &= b(Z,\lambda,\mu), \\ c &= c(Z,\lambda,\mu), \end{aligned}$$

This system can be inverted provided that we also match the square roots of the discriminants $\sqrt{\delta(Z,\lambda,\mu)} = \sqrt{\delta}$:

$$\begin{aligned} \lambda &= \lambda(a,b,c,\sqrt{\delta}), \\ \mu &= \mu(a,b,c,\sqrt{\delta}), \\ Z &= Z(a,b,c,\sqrt{\delta}). \end{aligned}$$

Using (1.6.28) the system above becomes:

$$\begin{aligned} aZ &= 8\lambda^3 + 12\lambda^2\mu + \frac{6\lambda\mu^2 + \mu^3}{1-Z}, \\ \frac{bZ}{3} &= -4\lambda^4 + \frac{6\lambda^2\mu^2 + 4\lambda\mu^3}{1-Z} + \frac{3\mu^4}{4(1-Z)^2}, \\ \frac{cZ}{3} &= 48\lambda^5 - \frac{40\lambda^3\mu^2}{1-Z} + \frac{15\lambda\mu^4 + 4\mu^5}{(1-Z)^2}. \end{aligned}$$

Amazingly, this nonlinear system can be solved *explicitly* (Klein [1], pp. 212-214). Skipping the technical details, we find the following

$$\begin{aligned} \lambda = \lambda(a,b,c,\sqrt{\delta}) &= \frac{(11a^3b + 2b^2c - ac^2) - a\sqrt{\delta}/(25\sqrt{5})}{24(a^4 - b^3 + abc)}, \\ Z = Z(a,b,c,\sqrt{\delta}) &= \frac{(48\lambda^2 a - 12\lambda b - c)^2}{64a^2(12\lambda(ac - b^2) - bc)} \\ \mu = \mu(a,b,c,\sqrt{\delta}) &= -\frac{96\lambda^3 a + 72\lambda^2 b + 6\lambda c - 12a^2 Z}{144\lambda^2 a + 12\lambda b + c}. \end{aligned}$$

With this we attain that *the zeros of the canonical quintic P and the zeros of the quintic resolvent P^* coincide as sets.* In (1.6.29) we described how the generators S and W of the icosahedral group act on the zeros of P^*. In order to obtain a root-by-root match we need to see how S and W act on the zeros of the canonical quintic P. This task will be carried out in the last subsection 1.6E.

D. Klein's Normalformsatz

Let k be a ground field containing the primitive d-th root of unity $\omega = e^{i\frac{2\pi}{d}}$, and let K/k be a Galois extension with cyclic Galois group C_d. Then, according to a result of Lagrange, there exists $Z \in k$ such that K is the splitting field of $\zeta^d - Z$, and K is generated by any of the zeros of this polynomial.

In view of the fact that $\zeta^d - Z = 0$ is the "polyhedral equation" for C_d, it is natural to look for a noncommutative analogue of this result, in which the field extension K of k is generated by any root of the icosahedral equation, and the Galois group of the field extension is $\mathcal{A}_5$. In order that the linear fractional transformations (1.2.23) that make up the icosahedral Möbius group I become k-automorphisms of the splitting field of the icosahedral equation (1.6.3), we need to assume that $\omega \in k$, where $\omega = e^{i\frac{2\pi}{5}}$.

Theorem 1.6.1. *Let k be a subfield of $\mathbf{C}$ that contains the primitive fifth root of unity $\omega = e^{i\frac{2\pi}{5}}$ and let $K \subset \mathbf{C}$ be a Galois extension of k with Galois group $\mathcal{A}_5$. Then, replacing k by a suitable quadratic extension, there exists $Z^* \in k$ such that K is generated by any solution ζ^* of the icosahedral equation (1.6.3) with parameter $Z^* = q(\zeta^*)$. Moreover, each solution ζ^* gives rise to an isomorphism $\phi : \mathcal{A}_5 \to I$ of the Galois group $\mathcal{A}_5$ to the icosahedral Möbius group I such that, if $\sigma \in \mathcal{A}_5$ is a k-automorphism of K that is mapped, under this isomorphism, to*

$$\phi(\sigma) : \zeta \mapsto \frac{a(\sigma)\zeta + b(\sigma)}{c(\sigma)\zeta + d(\sigma)}$$

then

$$\sigma^{-1}(\zeta^*) = \phi(\sigma)(\zeta^*) = \frac{a(\sigma)\zeta^* + b(\sigma)}{c(\sigma)\zeta^* + d(\sigma)}.$$

Remark 1. Theorem 1.6.1, the cornerstone of Klein's theory of the icosahedron, is called the "Normalformsatz." In 1861, Kronecker showed that the suitable quadratic extension k'/k ("akzessorische Irrationalität" as Klein called it) in the Normalformsatz cannot, in general, be dispensed with. As shown above, this extension comes in when reducing the general quintic to a canonical form (called "Hauptgleichung" in Klein [1]) by a Tschirnhaus transformation. For a modern account on this, see Serre [1]. A more elaborate account on the geometric theory of the general Tschirnhaus transformation, linear complexes, and the quintic is expounded in Klein [1].

Remark 2. Quadratic extensions do not change the setting in Theorem 1.6.1. In fact, if k' is a quadratic extension of the ground field k, then k' is not contained in K since the Galois group $\mathcal{A}_5$ of the extension K/k cannot contain any subgroup of index 2. We thus have $G(K \cdot k'/k') = G(K/k) = \mathcal{A}_5$.

Every irreducible quintic

$$P(\zeta) = \zeta^5 + a_1\zeta^4 + a_2\zeta^3 + a_3\zeta^2 + a_4\zeta + a_5, \quad a_1, \ldots, a_5 \in \mathbf{C}, \tag{1.6.31}$$

over $k(\ni \omega)$ with Galois group $\mathcal{A}_5$ has a splitting field K as in Theorem 1.6.1. Conversely, given K/k as in Theorem 1.6.1, there exists a quintic over k whose splitting field is K over k and whose Galois group is $\mathcal{A}_5$. Indeed, consider a subgroup of $\mathcal{A}_5$ isomorphic with $\mathcal{A}_4$. By abuse of notation, we denote this subgroup by $\mathcal{A}_4$. The field k is properly contained in the fixed field $K^{\mathcal{A}_4}$ so that there exists $\zeta_0 \in K^{\mathcal{A}_4} - k$. The $\mathcal{A}_5$-orbit of ζ_0 consists of 5 elements $\zeta_0, \ldots, \zeta_4$ since $\mathcal{A}_4$ is maximal in $\mathcal{A}_5$. Let $P(\zeta) = \prod_{j=0}^{4}(\zeta - \zeta_j)$ be the quintic resolvent associated to ζ_0. Then P is irreducible over k, K is the splitting field of P over k, and the Galois group $\mathcal{A}_5$ can be identified with the group of even permutations on the roots $\zeta_0, \ldots, \zeta_4$.

To prove Theorem 1.6.1, K is viewed as the splitting field of an irreducible quintic (with Galois group $\mathcal{A}_5$). The suitable quadratic extension $k(\sqrt{\delta})$ of the ground field k is due to the reduction of the quintic to canonical form by a Tschirnhaus transformation as discussed in Section 1.6/B.

E. Geometry of the Canonical Equation

We saw in Sections 1.6/B-C that the complex projective quadric $\mathcal{Q}_0$ in (1.6.30) parametrizes the points that correspond to the solutions (1.6.27) of the canonical resolvent of the icosahedral equation, and to the roots of the irreducible quintic (1.6.13) in canonical form (obtained from the general quintic by a Tschirnhaus transformation). In this final subsection, by "root-by-root matching" these two parametrizations, we indicate a proof of Theorem 1.6.1.

The geometry of the projective quadric $\mathcal{Q}_0 \subset \mathbf{C}P_0^3$ as a "doubly ruled surface" is well-known. In fact, the so-called Lagrange substitution, a linear equivalence between $\mathbf{C}P_0^3$ and $\mathbf{C}P^3$, transforms the defining equations $\sum_{j=0}^{4}\zeta_j = \sum_{j=0}^{4}\zeta_j^2 = 0$ of $\mathcal{Q}_0$ into the *single equation*

$$\xi_1\xi_4 + \xi_2\xi_3 = 0, \qquad [\xi_1 : \cdots : \xi_4] \in \mathbf{C}P^3.$$

The Lagrange substitution will be given explicitly below. The equation above defines the complex surface

$$\mathcal{Q} = \{[\xi_1 : \ldots : \xi_4] \in \mathbf{C}P^3 \,|\, \xi_1\xi_4 + \xi_2\xi_3 = 0\}$$

in $\mathbf{C}P^3$. We will identify $\mathcal{Q}_0$ with $\mathcal{Q}$ under this linear equivalence below.

For each value of a parameter $c^* \in \hat{\mathbf{C}}$, the equations

$$-\frac{\xi_1}{\xi_2} = \frac{\xi_3}{\xi_4} = c^* \tag{1.6.32}$$

define a complex projective line in $\mathcal{Q}$. We call this a *generating line of the first kind* (with parameter c^*). In a similar vein, for $c^{**} \in \hat{\mathbf{C}}$, the equations

$$\frac{\xi_1}{\xi_3} = -\frac{\xi_2}{\xi_4} = c^{**} \tag{1.6.33}$$

define in $\mathcal{Q}$ a *generating line of the second kind* (with parameter c^{**}). The two families of generating lines satisfy the following properties: (1) Each point of the quadric is the intersection of two generating lines of different kind; (2) Any two generating lines of different kind intersect at exactly one point; (3) Any two distinct generating lines of the same kind are disjoint.

Due to the linear equivalence between $\mathcal{Q}$ and $\mathcal{Q}_0$, the entire construction in $\mathcal{Q}$ can be carried over to our initial quadric $\mathcal{Q}_0$. Note that, by (1.6.32)-(1.6.33), c^* and c^{**} are quotients of linear forms in the variables $\zeta_0, \ldots, \zeta_4$ subject to (1.6.16).

If we fix a point $o \in \mathcal{Q}_0$ as the origin, then the generating lines of the first and second kind passing through o, denoted by $\mathbf{C}P^*$ and $\mathbf{C}P^{**}$, can be viewed as axes of a "rectilinear" coordinate system. With respect to this coordinate system, any point in $\mathcal{Q}_0$ can be uniquely represented by a pair of complex coordinates $(c^*, c^{**}) \in \mathbf{C}P^* \times \mathbf{C}P^{**}$ in an obvious manner. This gives a biholomorphic equivalence

$$\mathcal{Q}_0 = \mathbf{C}P^* \times \mathbf{C}P^{**}. \tag{1.6.34}$$

Recall that the symmetric group $\mathcal{S}_5$ acts on $\mathbf{C}P^4$ by permuting the homogeneous coordinates. In view of the symmetries in (1.6.16), this action leaves $\mathcal{Q}_0 \subset \mathbf{C}P_0^3$ invariant. From the point of view of projective geometry, $\mathcal{S}_5$ acts on $\mathcal{Q}_0$ by projective collineations, and thereby each collineation (corresponding to an element) in $\mathcal{S}_5$ maps generating lines to generating lines. By continuity, each collineation in $\mathcal{S}_5$ either maps the generating lines within a family to generating lines in the same family, or interchanges the generating lines between the two families. Let $\mathcal{G} \subset \mathcal{S}_5$ be the subgroup that preserves the generating lines in each family. We claim that $\mathcal{G} = \mathcal{A}_5$. It is clear that the index of $\mathcal{G}$ in $\mathcal{S}_5$ is at most 2. Since the alternating group $\mathcal{A}_5$ is the only index 2 subgroup in $\mathcal{S}_5$, it follows that $\mathcal{A}_5 \subset \mathcal{G}$. For the reverse inclusion, notice that $\mathcal{G}$ acts on $\mathbf{C}P^*$ by complex automorphisms, and this realizes $\mathcal{G}$ as a subgroup of $Aut\,(\mathbf{C}P^*)$. The choice of a nonhomogeneous coordinate in $\mathbf{C}P^*$ identifies $\mathbf{C}P^*$ with $\hat{\mathbf{C}}$, and $Aut\,(\mathbf{C}P^*)$ with the Möbius group $\mathcal{M}\,(\hat{\mathbf{C}})$. On the other hand, by Theorem 1.3.1, the largest finite subgroup of $\mathcal{M}\,(\hat{\mathbf{C}})$ is $\mathcal{A}_5$. The claim follows. In particular, we obtain that the collineations that correspond to the odd permutations in $\mathcal{S}_5$ interchange the two families of generating lines.

By the very definition of the equivalence (1.6.34), the action of $\mathcal{A}_5$ on the two families of generating lines induces an action of $\mathcal{A}_5$ on both $\mathbf{C}P^*$ and $\mathbf{C}P^{**}$ such that (1.6.34) is $\mathcal{A}_5$-equivariant with $\mathcal{A}_5$ acting on the product $\mathbf{C}P^* \times \mathbf{C}P^{**}$ diagonally.

As we saw above, the action of $\mathcal{A}_5$ on generating lines of the first and second kind realizes $\mathcal{A}_5$ as a subgroup of $Aut\,(\mathbf{C}P^*)$ and a subgroup of $Aut\,(\mathbf{C}P^{**})$. In a similar vein, a collineation of $\mathcal{Q}_0$ corresponding to an odd permutation in $\mathcal{S}_5$ gives rise to a holomorphic equivalence of $\mathbf{C}P^*$ and $\mathbf{C}P^{**}$. The (outer) automorphism of $\mathcal{A}_5$ by this odd permutation (within $\mathcal{S}_5$) then carries the action of $\mathcal{A}_5$ on $\mathbf{C}P^{**}$ into an action of $\mathcal{A}_5$ on $\mathbf{C}P^*$, and this latter action is equivalent to the original action of $\mathcal{A}_5$ on $\mathbf{C}P^*$ via complex automorphisms.

We now consider the projections $\pi^* : \mathcal{Q}_0 \to \mathbf{C}P^*$ and $\pi^{**} : \mathcal{Q}_0 \to \mathbf{C}P^{**}$. Let

$$\tilde{\mathcal{Q}}_0 = \left\{ (\zeta_0, \ldots, \zeta_4) \in \mathbf{C}^5 - \{0\} \;\middle|\; \sum_{j=0}^{4} \zeta_j = \sum_{j=0}^{4} \zeta_j^2 = 0 \right\}$$

and define $\pi : \tilde{\mathcal{Q}}_0 \to \mathcal{Q}_0$ to be the restriction of the canonical projection $\mathbf{C}^5 - \{0\} \to \mathbf{C}P^4$. We set

$$\zeta^* = \pi^* \circ \pi \text{ and } \zeta^{**} = \pi^{**} \circ \pi.$$

We take a closer look at ζ^*. A nonhomogeneous coordinate on $\mathbf{C}P^*$ identifies $\mathbf{C}P^*$ with $\hat{\mathbf{C}}$, and ζ^* can be viewed as the composition

$$(\mathbf{C}^5 - \{0\} \supset)\, \tilde{\mathcal{Q}}_0 \xrightarrow{\pi} \mathcal{Q}_0 \xrightarrow{\pi^*} \mathbf{C}P^* = \hat{\mathbf{C}}(\supset \mathbf{C}).$$

In view of the linear equivalence of $\mathcal{Q}_0$ and $\mathcal{Q}$, we see that ζ^* is a rational function in the variables $\zeta_0, \ldots, \zeta_4$ subjected to (1.6.16). By construction, $\mathcal{A}_5$ acts on these variables by even permutations and this induces an action of $\mathcal{A}_5$ on $\mathbf{C}P^*$ by complex automorphisms. With a choice of a nonhomogeneous coordinate on $\mathbf{C}P^*$, this latter action is by linear fractional transformations. This identifies $\mathcal{A}_5$ with a subgroup in $\mathcal{M}\,(\hat{\mathbf{C}})$. Different choices of nonhomogeneous coordinates on $\mathbf{C}P^*$ give rise to conjugate subgroups in $\mathcal{M}\,(\hat{\mathbf{C}})$. By Theorem 1.3.1, there is a nonhomogeneous coordinate on $\mathbf{C}P^*$ with respect to which $\mathcal{A}_5$ is identified with the icosahedral Möbius group I. From now on we assume that this choice has been made, and we let $\phi : \mathcal{A}_5 \to I$ denote the corresponding isomorphism. Summarizing, we see that $\zeta^* : \tilde{\mathcal{Q}}_0 \to \hat{\mathbf{C}}$ is ϕ-equivariant, where $\mathcal{A}_5$ acts on $\tilde{\mathcal{Q}}_0$ by permuting the coordinates, and I acts on $\hat{\mathbf{C}}$ as the icosahedral Möbius group.

When K is considered as the splitting field of a canonical quintic (1.6.13) with roots $\zeta_0, \ldots, \zeta_4$ and Galois group $\mathcal{A}_5$ then ζ^* becomes an element of $K = k(\zeta_0, \ldots, \zeta_4)$. The resolvent polynomial that ζ^* satisfies must be of degree 60 with coefficients a, b, c and $\sqrt{\delta}$, where δ is the discriminant of (1.6.13). Since ζ^* is ϕ-equivariant, the 60 roots of the resolvent polynomial are nothing but the icosahedral linear fractional transformations applied to ζ^*. By the proof of Theorem 1.3.1, the resolvent polynomial must be icosahedral. Thus, ζ^* satisfies the icosahedral equation

$$q(\zeta^*(\zeta_0, \ldots, \zeta_4)) = Z^*(a, b, c, \sqrt{\delta}),$$

where $\zeta_0, \ldots, \zeta_4$ are subjected to (1.6.16), and the parameter Z^* on the right-hand side depends on $a, b, c, \sqrt{\delta}$ rationally.

Remark. The situation is completely analogous for the generating lines of the second kind. We obtain a ϕ-equivariant rational function $\zeta^{**} : \mathbf{C}^5 \to \mathbf{C}$ that satisfies the icosahedral equation

$$q(\zeta^{**}(\zeta_0, \ldots, \zeta_4)) = Z^{**}(a, b, c, \sqrt{\delta}).$$

The parameters a, b, c are invariant under the entire symmetric group $\mathcal{S}_5$, while $\sqrt{\delta}$ changes its sign when the roots are subjected to odd permutations. Since the two actions of $\mathcal{A}_5$ on $\mathbf{C}P^*$ and $\mathbf{C}P^{**}$ are conjugate under the odd permutations in $\mathcal{S}_5$, we obtain

$$Z^*(a, b, c, -\sqrt{\delta}) = Z^{**}(a, b, c, \sqrt{\delta}).$$

We close this section by making our construction very explicit. First of all, for the stated linear equivalence between $\mathcal{Q}_0$ and $\mathcal{Q}$, we define $\iota : \mathbf{C}P^3 \to \mathbf{C}P^4$ by

$$\iota([\xi_1 : \ldots : \xi_4]) = [\zeta_0 : \ldots : \zeta_4],$$

where

$$\zeta_j = \sum_{l=1}^{4} \omega^{-jl} \xi_l, \quad j = 0, \ldots, 4.$$

Since $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$, we have

$$\sum_{j=0}^{4} \zeta_j = \sum_{l=1}^{4} \left(\sum_{j=0}^{4} \omega^{-jl} \right) \xi_l = 0.$$

Thus the linear map ι sends $\mathbf{C}P^3$ into the linear slice $\mathbf{C}P_0^3 \subset \mathbf{C}P^4$. Actually, ι is a linear isomorphism between $\mathbf{C}P^3$ and $\mathbf{C}P_0^3$. As simple computation shows, the inverse $\iota^{-1} : \mathbf{C}P_0^3 \to \mathbf{C}P^3$ is given by

$$\xi_l = \frac{1}{5} \sum_{j=0}^{4} \omega^{jl} \zeta_j, \quad l = 1, \ldots, 4.$$

To translate the defining equation $\sum_{j=0}^{4} \zeta_j^2 = 0$ of $\mathcal{Q}_0$ in terms of the ξ_l's, we compute

$$\begin{aligned} \sum_{j=0}^{4} \zeta_j^2 &= \sum_{j=0}^{4} \left(\sum_{l=1}^{4} \omega^{-jl} \xi_l \right)^2 \\ &= \sum_{l,l'=1}^{4} \left(\sum_{j=0}^{4} \omega^{-j(l+l')} \right) \xi_l \xi_{l'} \\ &= 10(\xi_1 \xi_4 + \xi_2 \xi_3). \end{aligned}$$

The last equality is because $\sum_{j=0}^{4} \omega^{-j(l+l')} = 5$ iff $l + l' = 5$, and zero otherwise. This shows that the quadrics $\mathcal{Q}$ and $\mathcal{Q}_0$ correspond to each other.

Given that the variables ξ_l, $l = 1, \ldots, 4$, are linear forms in ζ_j, $j = 0, \ldots, 4$, the equations for the generating lines give explicit rational dependence of ζ^* and ζ^{**} on ζ_j, $j = 0, \ldots, 4$. Due to our explicit formulas, we can determine the linear fractional transformations that ζ^* and ζ^{**} undergo when the ζ_j's are subjected to even permutations. We work this out for π^*. (The case of π^{**} can be treated analogously.) We will give explicit formulas only for the generators S and W of the icosahedral group. We claim that S (multiplication by ω) corresponds to the cyclic permutation

$$S : \zeta_j \mapsto \zeta_{j+1(\bmod 5)}, \quad j = 0, \ldots, 4.$$

Indeed, applying S, we have

$$\begin{aligned} c^* = -\frac{\xi_1}{\xi_2} &= -\frac{\sum_{j=0}^{4} \omega^j \zeta_j}{\sum_{j=0}^{4} \omega^{2j} \zeta_j} \mapsto -\frac{\sum_{j=0}^{4} \omega^j \zeta_{j+1(\bmod 5)}}{\sum_{j=0}^{4} \omega^{2j} \zeta_{j+1(\bmod 5)}} \\ &= -\frac{\sum_{j=0}^{4} \omega^{j-1} \zeta_j}{\sum_{j=0}^{4} \omega^{2(j-1)} \zeta_j} = -\frac{\omega \xi_1}{\xi_2} = \omega c^*, \end{aligned}$$

and the claim follows. Similarly, W corresponds to the permutation

$$W : \zeta_0 \mapsto \zeta_0, \ \zeta_1 \leftrightarrow \zeta_2, \ \zeta_3 \leftrightarrow \zeta_4.$$

The proof of this is tedious. Using the explicit form of W in (1.2.22) and (1.2.17), we have

$$Wc^* = -\frac{(\omega - \omega^4)\xi_1 + (\omega^2 - \omega^3)\xi_2}{(\omega^2 - \omega^3)\xi_1 - (\omega - \omega^4)\xi_2} = -\frac{\tau\xi_1 + \xi_2}{\xi_1 - \tau\xi_2},$$

where τ is the golden section. On the other hand, permuting the ζ_j's according to the recipe above, and rewriting the corresponding quotient in terms of the ξ_l's, we have

$$\begin{aligned} &-\frac{\zeta_0 + \omega^2\zeta_1 + \omega\zeta_2 + \omega^4\zeta_3 + \omega^3\zeta_4}{\zeta_0 + \omega^4\zeta_1 + \omega^2\zeta_2 + \omega^3\zeta_3 + \omega\zeta_4} \\ &= -\frac{(1+2\omega+2\omega^4)\xi_1 + (3+\omega^2+\omega^3)\xi_2 + (3+\omega+\omega^4)\xi_3 + (1+2\omega^2+2\omega^3)\xi_4}{(3+\omega^2+\omega^3)\xi_1 + (1+2\omega^2+2\omega^3)\xi_2 + (1+2\omega+2\omega^4)\xi_3 + (3+\omega+\omega^4)\xi_4} \\ &= -\frac{(1+2/\tau)\xi_1 + (3-\tau)\xi_2 + (3+1/\tau)\xi_3 + (1-2\tau)\xi_4}{(3-\tau)\xi_1 + (1-2\tau)\xi_2 + (1+2/\tau)\xi_3 + (3+1/\tau)\xi_4} \\ &= -\frac{\xi_1 + \xi_2/\tau + \xi_3\tau - \xi_4}{\xi_1/\tau - \xi_2 + \xi_3 + \xi_4\tau} \\ &= -\frac{(\tau\xi_1 + \xi_2)(1/\tau + \xi_3/\xi_1)}{(\xi_1 - \tau\xi_2)(1/\tau + \xi_3/\xi_1)} = -\frac{\tau\xi_1 + \xi_2}{\xi_1 - \tau\xi_2}. \end{aligned}$$

Here we used $1+2/\tau = \sqrt{5}$, $3-\tau = \sqrt{5}/\tau$, $3+1/\tau = \sqrt{5}\tau$, $1-2\tau = -\sqrt{5}$, and $\xi_1\xi_4 + \xi_2\xi_3 = 0$. The permutation rule for W follows.

Comparing how S and W transform the ζ_j's and the t_j's, we see that when ζ^* is subjected to the linear fractional transformations of the icosahedral group I then this action can be realized as even permutations on its variables *in exactly the same manner* as $\mathcal{A}_5$ acts on the roots $t_0, \ldots, t_4$ of the icosahedral resolvent. This means that an $\mathcal{A}_5$-equivariant one-to-one-correspondence $\zeta_j \leftrightarrow t_j$, $j = 0, \ldots, 4$, can be established between the two sets $\{\zeta_0, \ldots, \zeta_4\}$ and $\{t_0, \ldots, t_4\}$. (ζ_0 and t_0 are the unique fixed points of W, and $\zeta_j = S^j(\zeta_0)$ corresponds to $t_j = S^j(t_0)$, $j = 1, \ldots, 4$.)

We can get a closer look at the correspondence above by working out the generating lines *in terms of the roots* $\{t_0, \ldots, t_4\}$ *of the icosahedral resolvent.* In perfect analogy with the Lagrange substitutions, we put

$$t_j = \sum_{l=1}^{4} \omega^{-jl} X_l, \quad j = 0, \ldots, 4,$$

and

$$X_l = \frac{1}{5} \sum_{j=0}^{4} \omega^{jl} t_j, \quad l = 1, \ldots, 4.$$

To work out X_l, we use the explicit forms of Ξ_j and $\Omega_j \Xi_j$ in (1.6.19) and (1.6.22). Substituting them into the expression for t_j in (1.6.27) and using (1.6.26), we obtain

$$t_j = (\omega^{4j} z - \omega^{3j} w)A + (\omega^{2j} z + \omega^j w)B,$$

where A, B are linear in λ and μ. (Here z and w are the complex arguments of our forms with $\zeta = z/w$.) With this, the inverse of the Lagrange substitution becomes

$$\begin{aligned} X_l &= \frac{1}{5} \sum_{j=0}^{4} \omega^{jl} (\omega^{4j} z - \omega^{3j} w)A \\ &\quad + \frac{1}{5} \sum_{j=0}^{4} \omega^{jl} (\omega^{2j} z - \omega^j w)B \\ &= (\delta_{1l} z - \delta_{2l} w)A + (\delta_{3l} z + \delta_{4l} w)B, \end{aligned}$$

where we used the Kronecker delta function δ_{jl} (= 1 iff $j = l$ and zero otherwise). Writing the cases out, we have

$$X_1 = zA, \quad X_2 = -wA, \quad X_3 = zB, \quad X_4 = wB.$$

For the parameters C^* and C^{**} of the generating lines defined by

$$-\frac{X_1}{X_2} = \frac{X_3}{X_4} = C^*, \quad \frac{X_1}{X_3} = -\frac{X_2}{X_4} = C^{**},$$

we obtain

$$C^* = \frac{z}{w} = \zeta, \quad C^{**} = \frac{A}{B}.$$

Since ζ^* and ζ^{**} are essentially given by the projections $(C^*, C^{**}) \mapsto C^*$ and $(C^*, C^{**}) \mapsto C^{**}$, we thus have

$$\zeta^*(t_0, \ldots, t_4) = \zeta \text{ and } \zeta^{**}(t_0, \ldots, t_4) = \frac{A}{B}.$$

The first equation is enlightening. Once again writing the canonical resolvent in the short form

$$P^*(X) = \prod_{j=0}^{4}(X - t_j) = X^5 + 5a(Z, \lambda, \mu)X^2 + 5b(Z, \lambda, \mu)X + c(Z, \lambda, \mu),$$

we obtain

$$\begin{aligned} q(\zeta^*(t_0, \ldots, t_4)) &= Z^*(a(Z, \lambda, \mu), b(Z, \lambda, \mu), c(Z, \lambda, \mu), \sqrt{\delta(Z, \lambda, \mu)}) \\ &= q(\zeta) = Z. \end{aligned}$$

As noted above, the zeros of the canonical quintic P and the zeros of the quintic resolvent P^* can be made to coincide *as sets* by inverting a nonlinear system of equations explicitly. On the other hand, we also saw that there exists an $\mathcal{A}_5$-equivariant correspondence between these sets of roots. Imposing these we obtain

$$\zeta_j = t_j, \quad j = 0, \ldots, 4.$$

With this, we can now describe how to solve a given irreducible quintic. First we use the Tschirnhaus transformation to reduce the quintic to a canonical form $P(z) = z^5 + 5az^2 + 5bz + c$. This amounts to solving a quadratic equation. We also compute the discriminant δ from the explicit form given in (1.6.15). Then we substitute the coefficients a, b, c into the right-hand sides of the equations in the explicitly inverted system and obtain λ, μ and Z. We now solve the icosahedral equation for this particular value of Z to obtain ζ as a ratio of hypergeometric functions. By working out the forms Ω_j and Ξ_j using the particular values of λ, μ, Z we obtain

$$t_j = t_j(Z, \lambda, \mu), \quad j = 0, \ldots, 4.$$

Since $t_j = \zeta_j$, these are the five roots of our quintic. Tracing our steps back, we see that $\zeta_0, \ldots, \zeta_4$ depend rationally on ζ^*. In particular, when $K = k(\zeta_0, \ldots, \zeta_4)$ is the splitting field of the canonical quintic, we obtain that ζ^* generates K over k. This was the missing piece in the proof of Theorem 1.6.1.

Problems

1.1. Prove Theorem 1.1.1 using linear algebra as follows. Given a linear isometry S of $\mathbf{R}^3$, use the intermediate value theorem of calculus to show that the (cubic) characteristic polynomial $\det(S - \lambda I)$ always has a real root λ_0. Use the isometry property of S to obtain $\lambda_0 = \pm 1$. Conclude that an eigenvector $p_0 \in S^2$ corresponding to λ_0 satisfies $S(p_0) = \pm p_0$.

1.2. (a) Reformulate Theorem 1.1.1 as follows: Any special orthogonal matrix $S \in SO(3)$ can be diagonalized as $R_\theta \oplus [1]$, where

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

is the matrix of a plane rotation with angle θ.
(b) Generalize the proof of Theorem 1.1.1 to obtain an extension of (a) for any dimension.

1.3. (a) Show that the area of a spherical triangle with angles α, β, γ is $A = \alpha + \beta + \gamma - \pi$. (This is the "spherical excess formula" of Albert Girard in 1629.) (Hint: First derive a formula for the area of a spherical wedge. Second, consider the extension of the sides of a spherical triangle to great circles, and realize that these great circles subdivide S^2 into 8 spherical triangles. Third, use the area formula for various spherical wedges repeatedly.)
(b) Show that the area of a spherical n-sided polygon is the sum of its angles minus $(n-2)\pi$.
(c) Prove Euler's theorem for convex polyhedra using (b) as follows. Let P be a convex polyhedron. Place P inside S^2 such that P contains the origin in its interior. Project the boundary of P onto S^2 from the origin. Sum up *all* angles of the projected spherical graph with V vertices, E edges and F faces in two ways. First, counting the angles at each vertex, this sum is $2\pi V$. Second, count the angles for each face by converting the angle sum into spherical area, and use that the total area of S^2 is 4π.

1.4. (a) Given a finite set of points in $\mathbf{R}^3$, show that there is a unique smallest closed ball that contains all the points in this set.
(b) Use (a) to show that every finite group of isometries in $\mathbf{R}^3$ fixes at least one point. (Hint: Let the finite set of points in (a) be an orbit of the group. Prove that the minimal ball in (a) is left invariant by the group. Finally, show that the center of this ball is a fixed point.)

1.5. (a) Show that, in $\mathbf{R}^2$, the composition of two reflections in lines is a rotation or a translation, according to whether the two lines of reflection are intersecting or parallel. In the former case, the angle of the rotation is twice the angle between the lines; in the latter, the length of the translation is twice the distance between the lines.
(b) Show that, in $\mathbf{R}^3$, the composition of two reflections in planes is a rotation or a translation, according to whether the two planes are intersecting or parallel. In the former case, the angle of the rotation is twice the dihedral angle between the planes; in the latter, the length of the translation is twice the distance between the planes.

(c) By Theorem 1.1.1, in $\mathbf{R}^3$, the composition of two rotations with intersecting axes of rotation is another rotation. Use (b) to prove this directly. (Hint: Use the fact that a rotation is the composition of two reflections in planes containing the rotation axis. The choice of the planes is not unique, only the dihedral angle between them.)
(d) Show that two half-turns S_1 and S_2 with (distinct) intersecting axes commute ($S_1S_2 = S_2S_1$) iff the axes are perpendicular. Conclude that in this case the composition S_1S_2 is also a half-turn with axis being perpendicular to the axes of S_1 and S_2.

1.6. Generalize the notion of vertex figure to general convex polyhedra. Show that a convex polyhedron is regular iff its faces and vertex figures are regular (plane) polygons.

1.7. Show that the reciprocal of the side length of a regular decagon inscribed in the unit circle is the golden section. (Hint: Take a closer look at Figure 7.)

1.8. Let s and d be the side length and the diagonal length of a regular pentagon; $\tau = d/s$. Show that the side length and the diagonal length of the regular pentagon whose sides extend to the five diagonals of the original pentagon are $2s - d$ and $d - s$. Interpret this via paper folding. Conclude that τ is irrational.

1.9. Work out the rotational symmetries of the regular tetrahedron that correspond to all possible products of two transpositions.

1.10. (a) Label the vertices of the tetrahedron by the numbers 1,2,3,4. List the possible symmetries of the tetrahedron as permutations on $\{1, 2, 3, 4\}$.
(b) Based on the model of the colored octahedron as the intersection of a tetrahedron and its dual, make an explicit isomorphism between the color preserving (resp. reversing) symmetries of the octahedron and the even (resp. odd) permutations in $\mathcal{S}_4$.

1.11. Show that two golden cubes inscribed in a dodecahedron must have a common diagonal. (Hint: The five inscribed golden cubes have the total of 40 vertices, and the dodecahedron has 20 vertices.)

1.12. Prove that the symmetry group of the dodecahedron is simple (in the sense that it contains no proper normal subgroup), using the following argument. Let N be a normal subgroup of the symmetry group.
(a) Show that if N contains a rotation with axis through a vertex then N contains the rotations with axes through all the vertices of the dodecahedron.
(b) Derive similar statements for rotations with axes through the midpoint of the edges and the centroid of the faces.
(c) Counting the nontrivial rotations in N, conclude from (a)-(b) that $|N| = 1 + 24a + 20b + 15c$, where a, b, c are 0 or 1.
(d) Use the fact that $|N|$ divides 60 to show that either $a = b = c = 0$ or $a = b = c = 1$.

1.13. (a) Use the construction in the roof proof to obtain coordinates for the vertices of a dodecahedron in $\mathbf{R}^3$.
(b) Use the Pacioli model to obtain coordinates for the vertices of the icosahedron in $\mathbf{R}^3$.

1.14. Construct a golden rectangle with straightedge and compass.

1.15. By the famous four color theorem, every graph imbedded in S^2 (no two edges cross over) can be four colored in the sense that each vertex of the graph receives one of the four colors, and no two vertices connected by an edge receive the same color. Derive a four coloring of the vertices of the dodecahedron starting with the 5 coloring of the icosahedral dual described in the text. (Notice that a fifth color can be deleted from the five coloring of the icosahedron by allowing faces that touch only at a vertex to receive the same color.)

1.16. Place an icosahedron in $\mathbf{R}^3$ such that the three inscribed golden rectangles are contained in the three coordinate planes.
(a) Work out the matrix of an order 5 rotation that permutes the five disjoint color groups.
(b) Using (a) determine the coordinates of the vertices of the 5 circumscribed tetrahedra.

1.17. Prove the following:
(a) The extended symmetry group of the regular pyramid with regular n-gonal base P_n is $D_n \cdot C_n$.
(b) The extended symmetry group of the regular prism with regular n-gonal base P_n is $D_n \times C_2$, for n even, and $D_{2n} \cdot D_n$, for n odd.

1.18. Show that the dihedral group $D_2 \cong C_2 \times C_2$ is the symmetry group of a rectangle in $\mathbf{R}^3$.

1.19. Prove that every finite subgroup of $SO(3)$ is generated by one or two elements.

1.20. Show that the linear fractional transformations of the extended complex plane form a group under composition and inverse.

1.21. Show that associating to the matrix (1.2.3) the linear fractional transformation (1.2.1) defines a homomorphism of $SL(2, \mathbf{C})$ onto $\mathcal{M}(\hat{\mathbf{C}})$.

1.22. Verify that (1.2.5) and (1.2.6) are inverses of each other.

1.23. Perform the computations leading to the elements of the tetrahedral Möbius group in (1.2.14).

1.24. Prove that any finite subgroup of $\mathbf{C} - \{0\}$ is a cyclic subgroup of $S^1 \subset \mathbf{C} - \{0\}$. (Hint: The absolute values of the group elements form a multiplicative subgroup in $\mathbf{R} - \{0\}$.)

1.25. Derive the identity $h(-x) = -1/\overline{h(x)}$, $x \in S^2$, for the stereographic projection h.

1.26. Verify the following identities:

$$\begin{aligned} \mathit{Hess}\,(\Phi) &= -48\sqrt{3}i\Psi \\ \mathit{Hess}\,(\Psi) &= 48\sqrt{3}i\Phi \\ \mathit{Jac}\,(\Phi, \Psi) &= 32\sqrt{3}i\Omega \\ \mathit{Hess}\,(\Omega) &= -25\Phi\Psi \\ \mathit{Jac}\,(\Omega, \Phi) &= -4\Psi^2 \\ \mathit{Jac}\,(\Omega, \Psi) &= -4\Phi^2. \end{aligned}$$

1.27. Let $f : M \to N$ be a holomorphic n-fold branched covering between compact Riemann surfaces, and assume that M has genus p and N has genus q. Prove the *Riemann-Hurwitz relation*

$$p = n(q-1) + 1 + B/2,$$

where B is the total branch number, the sum of all branch numbers. (Hint: Each point in N is assumed precisely n times on M by f, counting multiplicities. At a branch value, this means that n is equal to the sum of all [branch number plus 1]'s, where the sum is over those branch points that map to the given branch value. Triangulate N such that every branch value is a vertex of the triangulation. Let V, E, and F denote the number of vertices, edges, and faces of this triangulation. Now pull the triangulation on N back to a triangulation on M via f. Prove that the induced triangulation on M has $nV - B$ vertices, nE edges and nF faces. Express the Euler characteristics $\chi(M) = 2 - 2p$ and $\chi(N) = 2 - 2q$ of M and N in terms of the vertices, edges, and faces of the triangulations and compare.)

1.28. Use (1.4.2) and the complex form of the Laplacian on $\mathbf{R}^4 = \mathbf{C}^2$ to prove that the components of the orbit map $f_\xi : \mathbf{R}^4 \to W_p$ are harmonic.

1.29. According to Schur's orthogonality relations in representation theory, the matrix elements of an irreducible representation are L_2-orthogonal with the same norm. Use this to show that any p-eigenmap $f_\xi : S^3 \to S_{R_p}$, $p = 2d$, obtained from the equivariant construction has L_2-orthonormal components (up to a suitable scaling of the scalar product on R_p). Prove a similar statement for $f_\xi : S^3 \to S_{W_p}$.

1.30. **(a)** Let V be a complex vector space, and $\langle\,,\,\rangle$ and $\langle\,,\,\rangle'$ two hermitian scalar products on V. Show that the linear endomorphism $A : V \to V$ satisfying

$$\langle u, v\rangle' = \langle Au, v\rangle, \quad u, v \in V,$$

is well-defined, invertible, and hermitian symmetric.
(b) Let $G \subset SO(V)$ be a subgroup and assume that both $\langle\,,\,\rangle$ and $\langle\,,\,\rangle'$ are G-invariant. Prove that A is a G-module endomorphism of V.
(c) Let G be as in (b) and assume that V is irreducible as a G-module. Use Schur's lemma to show that A is a real constant multiple of the identity. Conclude that, up to a real constant multiple, there is a unique G-invariant hermitian scalar product on V.

1.31. **(a)** Let $\rho_{Hopf} : SU(2) \to SO(3)$ be the homomorphism associated to the $SU(2)$-equivariance of the Hopf map in Example 1.4.2. Show that ρ_{Hopf} is the double cover $SU(2) \to SU(2)/\{\pm I\} = SO(3)$ (Corollary 1.2.2) precomposed by a quarter-turn around the second axis. (Hint: This follows from (1.2.10) by noticing that the rotation R that corresponds to $z + jw \in S^3$ via (1.2.8)-(1.2.9) sends the north pole N to the components of the Hopf map in reversed order.)

1.32. Show that U, V, W act on the octahedral forms Ω_j, $j = 0, \ldots, 4$, as follows

$$\begin{aligned} U &: \ \Omega_0 \mapsto \Omega_0,\ \Omega_1 \leftrightarrow \Omega_4,\ \Omega_2 \leftrightarrow \Omega_3 \\ V &: \ \Omega_0 \mapsto \Omega_0,\ \Omega_1 \leftrightarrow \Omega_3,\ \Omega_2 \leftrightarrow \Omega_4 \\ W &: \ \Omega_0 \mapsto \Omega_0,\ \Omega_1 \leftrightarrow \Omega_2,\ \Omega_3 \leftrightarrow \Omega_4. \end{aligned}$$

1.33. Interpret the zeros of the degree 8 forms Ξ_j, $j = 0, \ldots, 4$, geometrically.

1.34. (a) Work out the resolvent of the five octahedral forms using the following steps: First consider the product

$$\prod_{j=0}^{4}(\mathcal{X}-\Omega_j)=\mathcal{X}^5+a_1\mathcal{X}^4+a_2\mathcal{X}^3+a_3\mathcal{X}^2+a_4\mathcal{X}+a_5,$$

where $\mathcal{X}$ is used as variable. Using the description of the invariants of the icosahedral group in (1.3.15)-(1.3.17), conclude that this quintic resolvent reduces to

$$\mathcal{X}^5-10\mathcal{I}\mathcal{X}^3+45\mathcal{I}^2\mathcal{X}-\mathcal{J}.$$

Introduce the new variable

$$X=12\frac{\mathcal{I}^2}{\mathcal{J}}\mathcal{X}$$

(depending only on $\zeta=z/w$) and use the second table in Section 1.3 to arrive at the quintic icosahedral resolvent polynomial

$$\begin{aligned}P^*(X)&=\prod_{j=0}^{4}(X-r_j)\\&=X^5-\frac{5}{6(1-Z)}X^3+\frac{5}{16(1-Z)^2}X-\frac{1}{12(1-Z)^2}.\end{aligned}$$

(b) Describe the geometry of the "solution set" $(r_0(\zeta),\dots,r_4(\zeta))\in\mathbf{C}^5$ as Z varies in $\hat{\mathbf{C}}$ and $q(Z)=\zeta$ as follows. Introduce the sums of various powers

$$\sigma_m=\sum_{j=0}^{4}r_j^m,\quad m=1,\dots,4.$$

and verify

$$\sigma_1=0,\ \sigma_2=\frac{5}{3(1-Z)},\ \sigma_3=0,\ \sigma_4=\frac{5}{36(1-Z)^2}.$$

Notice that $\sigma_2^2=20\sigma_4$. Define

$$\mathcal{D}=\left\{[r_0:\dots:r_4]\in\mathbf{C}P^4\;\middle|\;\sum_{j=0}^{4}r_j=\sum_{j=0}^{4}r_j^3=0\right\}$$

and

$$\mathcal{F}=\left\{[r_0:\dots:r_4]\in\mathbf{C}P^4\;\middle|\;\sum_{j=0}^{4}r_j=0,\ \left(\sum_{j=0}^{4}r_j^2\right)^2=20\sum_{j=0}^{4}r_j^4\right\},$$

and prove that $\mathcal{D}$ and $\mathcal{F}$ are smooth algebraic surfaces in $\mathbf{C}P^3$ of degree 3 and 4, respectively (where $\mathbf{C}P^3$ is identified with the linear slice of $\mathbf{C}P^4$ defined by $\sum_{j=0}^{4}r_j=0$). Show that the projective points that correspond to the solutions of the quintic icosahedral resolvent above fill the smooth algebraic curve $\mathcal{D}\cap\mathcal{F}\subset\mathbf{C}P^3$ of degree 12. (For a detailed treatment, see Fricke [1].)

1.35. (a) Derive an icosahedral resolvent polynomial of degree 6 based on the six quartic forms that vanish on the six pairs of antipodal vertices of the icosahedron

as follows. Set $\phi_\infty = 5z^2w^2$, and apply the icosahedral substitutions WS^j to ϕ_∞ to obtain

$$\phi_j(z,w) = (\omega^j z^2 + zw - \omega^{4j} w^2)^2, \quad j = 0, \ldots, 4.$$

Comparing coefficients, show that the resolvent form satisfies

$$\phi^6 - 10\mathcal{I}\phi^3 + \mathcal{H}\phi + 5\mathcal{I}^2 = 0.$$

(b) Work out the action of the icosahedral generators S and W on ϕ_∞, and ϕ_j, $j = 0, \ldots, 4$, and verify

$$\begin{aligned} S: &\ \phi_\infty \mapsto \phi_\infty, \ \ \phi_j \mapsto \phi_{j+1(\bmod 5)}, \ \ j = 0, \ldots, 4; \\ W: &\ \phi_\infty \leftrightarrow \phi_0, \ \ \phi_1 \leftrightarrow \phi_4, \ \ \phi_2 \leftrightarrow \phi_3. \end{aligned}$$

Interpret these transformation rules as congruences

$$\begin{aligned} S: &\ j' \equiv j + 1 (\bmod 5); \\ W: &\ j' \equiv -\frac{1}{j} (\bmod 5). \end{aligned}$$

Show that the icosahedral group (the Galois group of this resolvent) is isomorphic with $PSL(2, \mathbf{Z}_5)$ given by

$$j \mapsto j' \equiv -\frac{aj+b}{cj+d} (\bmod 5), \ \ ad - bc \equiv 1 (\bmod 5), \ \ a, b, c, d \in \mathbf{Z}.$$

2
Moduli for Eigenmaps

2.1 Spherical Harmonics

In this introductory section we summarize some basic concepts and facts on spherical harmonics. The general references are Vilenkin [1] and Berger-Gauduchon-Mazet [1].

Given a function $\xi : \mathbf{R}^{m+1} \to \mathbf{R}$, the Euclidean Laplacian $\triangle = \sum_{i=0}^{m} \partial_i^2$ and the spherical Laplacian $\triangle^{rS^m}$ of the sphere $rS^m = \{x \in \mathbf{R}^{m+1} \,|\, |x| = r\}$, $r > 0$, relate as

$$\triangle^{rS^m}(\xi|_{rS^m}) = \left(-\triangle\xi + \frac{1}{r^2}\partial_x^2\xi + \frac{m-1}{r^2}\partial_x\xi\right)\Big|_{rS^m}.$$

Here $\partial_i = \partial/\partial x_i$ is partial differentiation with respect to the i-th variable x_i, $i = 0, \ldots, m$, and $\partial_x = \sum_{i=0}^{m} x_i\partial_i$ is *radial differentiation.* (For details and a proof, see also Appendix 2.) In particular, if χ is a harmonic homogeneous polynomial of degree p, then $\partial_x\chi = p\chi$ and $\partial_x^2\chi = p^2\chi$, and we have

$$\triangle^{S^m}(\chi|_{S^m}) = p(p+m-1)\,\chi|_{S^m}.$$

This shows that the restriction $\chi|_{S^m}$ is an eigenfunction of $\triangle^{S^m}$ with eigenvalue

$$\lambda_p = \lambda_{m,p} = p(p+m-1).$$

Unless it is relevant, the dependence of the various geometric objects on m will be suppressed. This will not lead to confusion since most of the time, the dimension of the domain will be fixed.

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In the following, we assume that

$$m \geq 2 \quad \text{and} \quad p \geq 1.$$

Since $p = 1$ corresponds to linear functions, this case will be omitted from most considerations.

Let $\mathcal{P}^p = \mathcal{P}^p_{m+1} \subset \mathbf{R}[x_0, \ldots, x_m]$ denote the (real) vector space of homogeneous polynomials of degree p in the $(m+1)$ variables $x_0, \ldots, x_m$. Let $\mathcal{H}^p = \mathcal{H}^p_m \subset \mathcal{P}^p$ be the linear subspace of *harmonic* homogeneous polynomials of degree p. As we saw above, the restrictions $\chi|_{S^m}$, $\chi \in \mathcal{H}^p$, are eigenfunctions of $\triangle^{S^m}$ with eigenvalue λ_p. We will prove shortly that $\mathcal{H}^p|_{S^m} = \{\, \chi|_{S^m} \mid \chi \in \mathcal{H}^p \,\}$ is the entire λ_p-eigenspace. We will be primarily interested in $\chi|_{S^m}$ and not $\chi \in \mathcal{H}^p$, and this explains the lower index m (and not $m+1$) in $\mathcal{H}^p_m$. $\mathcal{P}^p$ carries an inner product, the *L^2-scalar product* on the restrictions to S^m:

$$\langle \xi_1, \xi_2 \rangle_{L^2} = \int_{S^m} \xi_1 \xi_2 \, v_{S^m}, \quad \xi_1, \xi_2 \in \mathcal{P}^p,$$

where v_{S^m} is the *volume form* on S^m.

Let

$$\rho^2 = |x|^2 = \sum_{i=0}^{m} x_i^2.$$

We have the orthogonal decomposition

$$\mathcal{P}^p = \mathcal{H}^p \oplus \mathcal{P}^{p-2} \cdot \rho^2, \quad p \geq 2. \tag{2.1.1}$$

To prove this, we first show the following:

Lemma 2.1.1. *The Laplacian $\triangle$ is injective on $\mathcal{P}^{p-2} \cdot \rho^2$.*

First let us derive a computational formula that will be used in several instances. For $\xi \in \mathcal{P}^q$, we claim that

$$\triangle(\xi \rho^{2k}) = 2k(2q + 2k + m - 1)\xi \rho^{2(k-1)} + \triangle\xi \cdot \rho^{2k}. \tag{2.1.2}$$

This is a simple computation in the use of the differentiation rule

$$\partial_i \rho^{2k} = 2k \rho^{2(k-1)} x_i, \quad i = 0, \ldots, m,$$

and homogeneity

$$\partial_x \xi = \sum_{i=0}^{m} x_i \partial_i \xi = q\xi.$$

Indeed, we have

$$\begin{aligned} \triangle(\xi \rho^{2k}) &= \sum_{i=0}^{m} \partial_i^2 (\xi \rho^{2k}) \\ &= \sum_{i=0}^{m} \partial_i (\rho^{2k} \partial_i \xi + 2k \rho^{2(k-1)} x_i \xi) \end{aligned}$$

$$
\begin{aligned}
&= 4k\rho^{2(k-1)} \sum_{i=0}^{m} x_i \partial_i \xi + \rho^{2k} \triangle \xi + 4k(k-1)\rho^{2(k-2)} \sum_{i=0}^{m} x_i^2 \xi \\
&+ 2k(m+1)\rho^{2(k-1)} \xi \\
&= 2k(2q + 2(k-1) + m + 1)\rho^{2(k-1)} \xi + \rho^{2k} \triangle \xi.
\end{aligned}
$$

(2.1.2) follows.

PROOF OF LEMMA 2.1.1. We let $\eta \in \mathcal{P}^{p-2}$. We need to work out $\triangle(\eta\rho^2)$. We write $\eta = \xi\rho^{2l}$, $l \geq 0$, where $\xi \in \mathcal{P}^{p-2(l+1)}$ is not divisible by ρ^2. Using (2.1.2) with $q = p - 2(l+1)$ and $k = l+1$, we have

$$
\begin{aligned}
\triangle(\eta\rho^2) &= \triangle(\xi\rho^{2(l+1)}) \\
&= 2(l+1)(2p - 2l + m - 3)\xi\rho^{2l} + (\triangle\xi) \cdot \rho^{2(l+1)}.
\end{aligned}
$$

If this were zero, then ξ would be divisible by ρ^2. This is a contradiction.

Returning to the proof of decomposition (2.1.1), we first note that the Laplacian maps $\mathcal{P}^p$ into $\mathcal{P}^{p-2}$ with kernel $\mathcal{H}^p$. By Lemma 2.1.1, $\mathcal{H}^p \cap \mathcal{P}^{p-2}\rho^2 = 0$. In particular

$$
\dim \mathcal{H}^p + \dim \mathcal{P}^{p-2} \leq \dim \mathcal{P}^p.
$$

On the other hand, since $\triangle : \mathcal{P}^p \to \mathcal{P}^{p-2}$, we have

$$
\dim \mathcal{P}^p - \dim \mathcal{H}^p \leq \dim \mathcal{P}^{p-2}.
$$

Combining these we obtain that the inequalities are actually equalities and

$$
\mathcal{P}^p = \mathcal{H}^p + \mathcal{P}^{p-2} \cdot \rho^2 \tag{2.1.3}
$$

is a direct sum. Iterating (2.1.3), we get the direct sum:

$$
\mathcal{P}^p = \sum_{j=0}^{[p/2]} \mathcal{H}^{p-2j} \rho^{2j}. \tag{2.1.4}
$$

Harmonic homogeneous polynomials of different degree are orthogonal since they are eigenfunctions of the (self-adjoint) Laplacian with different eigenvalues. Retracing our steps, we see that the direct sum (2.1.4) is orthogonal; in particular, we have (2.1.1).

Counting the number of monomials of degree p in $m+1$ variables, an easy induction (with respect to m) shows that

$$
\dim \mathcal{P}^p = \binom{m+p}{m}.
$$

Hence, by (2.1.1), we have

$$
\begin{aligned}
\dim \mathcal{H}^p &= \binom{p+m}{m} - \binom{p+m-2}{m} \qquad (2.1.5) \\
&= (2p+m-1)\frac{(p+m-2)!}{p!(m-1)!}.
\end{aligned}
$$

By (2.1.4), for $\xi \in \mathcal{P}^p$, there exist unique $\xi_j \in \mathcal{H}^{p-2j}$, $j = 0, \ldots, [p/2]$, such that

$$\xi = \sum_{j=0}^{[p/2]} \xi_j \rho^{2j}.$$

This is called the *canonical decomposition of* ξ *with coefficients* ξ_j, $j = 0, \ldots, [p/2]$.

We also see that all the eigenfunctions of the spherical Laplacian arise as restrictions of harmonic homogeneous polynomials to S^m, i.e., given an eigenfunction χ of $\triangle^{S^m}$ with eigenvalue λ then $\lambda = \lambda_p$ for some p and χ is the restriction to S^m of a harmonic homogeneous polynomial χ (denoted by the same symbol) of degree p. Indeed, this follows from the decomposition in (2.1.4) along with the Stone-Weierstrass theorem asserting that the (restrictions of) homogeneous polynomials form a dense set in the space of C^∞-functions on S^m.

If χ is an eigenfunction of $\triangle^{S^m}$ with eigenvalue λ_p, we say that χ is a *spherical harmonic of order* p. By abuse of terminology, we will not distinguish between a harmonic homogeneous polynomial χ and its restriction $\chi|_{S^m}$ as a spherical harmonic. In particular, a spherical harmonic of order 1 on S^m will be identified with a linear functional on $\mathbf{R}^{m+1}$: $\mathcal{H}^1_m = (\mathbf{R}^{m+1})^*$.

The idea of the proof of Lemma 2.1.1 can be used to prove the following:

Lemma 2.1.2. *Let* c_0 *and* c_1 *be positive constants and* $\eta \in \mathcal{P}^p$. *Then there exists a unique* $\xi \in \mathcal{P}^p$ *such that*

$$c_0 \xi + c_1 \triangle \xi \cdot \rho^2 = \eta. \tag{2.1.6}$$

PROOF. We consider the canonical decompositions

$$\xi = \sum_{j=0}^{[p/2]} \xi_j \rho^{2j} \quad \text{and} \quad \eta = \sum_{j=0}^{[p/2]} \eta_j \rho^{2j}.$$

Substituting, equation (2.1.6) can be easily resolved for ξ_j in terms of η_j. In fact, we have

$$\xi_j = \frac{\eta_j}{c_0 + 2j(2p - 2j + m - 1)c_1}.$$

The lemma follows.

The *harmonic projection operator* H is defined as the orthogonal projection $H : \mathcal{P}^p \to \mathcal{H}^p$. By the discussion above, H has kernel $\mathcal{P}^{p-2}\rho^2$. In terms of (the powers of) the Laplacian, for $\xi \in \mathcal{P}^p$, $H(\xi)$ is given by

$$H(\xi) = \xi + \sum_{j=1}^{[p/2]} \frac{(-1)^j (p-1)\ldots(p-j)}{j! \lambda_{2(p-1)} \cdots \lambda_{2(p-j)}} \triangle^j \xi \cdot \rho^{2j}. \tag{2.1.7}$$

To prove (2.1.7) we write

$$H(\xi) = \xi + \sum_{j=1}^{[p/2]} c_j \triangle^j \xi \cdot \rho^{2j},$$

where the coefficients c_j, $j = 1, \ldots, [p/2]$, are to be determined. Taking the Laplacian of both sides of this equation and using (2.1.2), we obtain

$$\begin{aligned} \triangle H(\xi) &= \triangle \xi + \sum_{j=1}^{[p/2]} c_j \triangle (\triangle^j \xi \cdot \rho^{2j}) \\ &= \triangle \xi \\ &\quad + \sum_{j=1}^{[p/2]} c_j (2j(2p - 2j + m - 1)\rho^{2(j-1)} \triangle^j \xi + \rho^{2j} \triangle^{j+1} \xi) = 0 \end{aligned}$$

The corresponding recurrence relation for the coefficients is

$$\begin{aligned} c_1 &= -\frac{1}{2(2p + m - 3)} = \frac{p-1}{\lambda_{2(p-1)}} \\ c_{j+1} &= -\frac{1}{2(j+1)(2p - 2j + m - 1)} c_j = -\frac{p-j}{(j+1)\lambda_{2(p-j)}} c_j, \quad j \geq 1. \end{aligned}$$

This gives (2.1.7).

We point out two consequences of the harmonic projection formula (2.1.7). First, we let $\xi = a^* \cdot \chi$, $\chi \in \mathcal{H}^p$, where $a^* \in \mathcal{H}^1 = (\mathbf{R}^{m+1})^*$ is the linear functional associated to $a \in \mathbf{R}^{m+1}$:

$$a^*(x) = \langle a, x \rangle, \quad x \in \mathbf{R}^{m+1}.$$

Since

$$\triangle \xi = \triangle (a^* \cdot \chi) = 2 \sum_{i=0}^{m} \partial_i a^* \cdot \partial_i \chi = 2 \sum_{i=0}^{m} a_i \partial_i \chi = 2 \partial_a \chi,$$

we obtain

$$H(a^* \cdot \chi) = a^* \cdot \chi - \frac{2p}{\lambda_{2p}} \partial_a \chi \cdot \rho^2, \quad a \in \mathbf{R}^{m+1}. \tag{2.1.8}$$

In particular, if $a = e_i$, the i-th vector in the standard basis $\{e_i\}_{i=0}^m \subset \mathbf{R}^{m+1}$, then $a^* = e_i^* = x_i$, and we get

$$H(x_i \chi) = x_i \chi - \frac{\rho^2}{2p + m - 1} \partial_i \chi, \quad i = 0, \ldots, m.$$

Second, for $\xi = x_m^p$, we have

$$\triangle^j (x_m^p) = p(p-1) \ldots (p - 2j + 1) x_m^{p-2j}, \quad j = 0, \ldots, [p/2].$$

Substituting this into (2.1.7), we obtain

$$H(x_m^p) = x_m^p \tag{2.1.9}$$

$$+\sum_{j=1}^{[p/2]} \frac{(-1)^j(p-1)\dots(p-j)p(p-1)\dots(p-2j+1)}{j!\lambda_{2(p-1)}\cdots\lambda_{2(p-j)}} x_m^{p-2j}\rho^{2j}.$$

The coefficient of $x_m^{p-2j}\rho^{2j}$ in the sum can be written in terms of the Gamma function Γ as follows

$$\begin{aligned}
&\frac{(-1)^j(p-1)\dots(p-j)p(p-1)\dots(p-2j+1)}{j!\lambda_{2(p-1)}\cdots\lambda_{2(p-j)}} \\
&= \frac{(-1)^j p(p-1)\dots(p-2j+1)}{j!2^j(2p+m-3)\dots(2p+m-2j-1)} \\
&= \frac{(-1)^j p!}{j!2^{2j}(p-2j)!\left(p+\frac{m-1}{2}-1\right)\dots\left(p+\frac{m-1}{2}-j\right)} \\
&= \frac{p!}{2^p\Gamma\left(p+\frac{m-1}{2}\right)} \frac{(-1)^j\Gamma\left(p+\frac{m-1}{2}-j\right)}{j!(p-2j)!} 2^{p-2j}.
\end{aligned}$$

With this (2.1.9) becomes

$$H(x_m^p) = \frac{p!}{2^p\Gamma\left(p+\frac{m-1}{2}\right)} \sum_{j=0}^{[p/2]} \frac{(-1)^j\Gamma\left(p+\frac{m-1}{2}-j\right)}{j!(p-2j)!} (2x_m)^{p-2j}\rho^{2j}.$$

We now introduce the *ultraspherical* (or *Gegenbauer*) *polynomial* C_d^a defined by

$$C_d^a(z) = \sum_{j=0}^{[d/2]} \frac{(-1)^j\Gamma(d+a-j)}{\Gamma(a)j!(d-2j)!}(2z)^{d-2j} \tag{2.1.10}$$

(cf. Szegö [1], p. 84, or Vilenkin [1], p. 458).

Comparing this with the expansion of $H(x_m^p)$ above, we obtain

$$H(x_m^p) = \frac{p!\Gamma\left(\frac{m-1}{2}\right)}{2^p\Gamma\left(p+\frac{m-1}{2}\right)} \rho^p C_p^{(m-1)/2}\left(\frac{x_m}{\rho}\right). \tag{2.1.11}$$

The ultraspherical polynomials are special cases of Jacobi polynomials, and they possess many important properties. The most important is orthonormality. Namely, for fixed a, the normalized ultraspherical polynomials

$$2^{a-1}\Gamma(a)\left(\frac{2(d+a)d!}{\pi\Gamma(d+2a)}\right)^{1/2} C_d^a(t), \quad d \geq 0,$$

form an orthonormal system on the interval $[-1,1]$ relative to the weight $(1-t^2)^{a-1/2}$:

$$\int_{-1}^{1} C_d^a(t)C_{d'}^a(t)(1-t^2)^{a-1/2}dt = 0, \quad d \neq d', \tag{2.1.12}$$

and

$$\int_{-1}^{1} C_d^a(t)^2(1-t^2)^{a-1/2}dt = \frac{\pi\Gamma(d+2a)}{2^{2a-1}\Gamma(a)^2(d+a)d!}. \tag{2.1.13}$$

(For a integral or half-integral, (2.1.12) follows from L^2-orthogonality of $H(x_{2a+1}^d)$ and $H(x_{2a+1}^{d'})$, $d \neq d'$; see Problem 2.22. A more general approach will be given in Section 2.8. The proof of (2.1.13) is more involved; two different approaches are found in Szegö [1], p. 67, via the Rodrigues formula for Jacobi polynomials, and in Vilenkin [1] via Schur's orthogonality relations applied to $\mathcal{H}^p$.)

$\mathcal{P}^p$ is the representation space of the orthogonal group $O(m+1)$, where the action of an isometry $g \in O(m+1)$ is given by $g : \xi \mapsto \xi \circ g^{-1}$, $\xi \in \mathcal{P}^p$. The volume form v_{S^m} is invariant under the usual action of $O(m+1)$ on S^m by isometries. Thus the L_2-scalar product on $\mathcal{P}^p$ is also $O(m+1)$-invariant. Since the Laplacian commutes with isometries (the elements of $O(m+1)$), and ρ^2 is $O(m+1)$-invariant, (2.1.4) gives a decomposition of $\mathcal{P}^p$ into the invariant subspaces $\mathcal{H}^{p-2j}\rho^{2j}$, $j = 0, \ldots, [p/2]$. When we regard $\mathcal{H}^{p-2j}\rho^{2j}$ *as an* $O(m+1)$*-submodule of* $\mathcal{P}^p$, *we suppress* ρ^{2j}, i.e., we write

$$\mathcal{P}^p = \sum_{j=0}^{[p/2]} \mathcal{H}^{p-2j}, \tag{2.1.14}$$

as $O(m+1)$-modules. By (2.1.7), H also commutes with the actions of $O(m+1)$ on $\mathcal{P}^p$ and on $\mathcal{H}^p$.

We now claim that (2.1.14) is the *irreducible* decomposition of $\mathcal{P}^p$ into $SO(m+1)$-submodules. To show this, we first prove that, up to constant multiple, $H(x_m^p)$ in (2.1.9) is the unique *zonal* in $\mathcal{H}^p$, i.e., a spherical harmonic left fixed by the subgroup $SO(m) = SO(m) \oplus [1] \subset SO(m+1)$. (In what follows, $SO(m)$ as a subgroup of $SO(m+1)$ will always mean $SO(m) \oplus [1]$.) This can be seen as follows. Let χ be a zonal in $\mathcal{H}^p$. Since $SO(m)$ acts on the tangent space $T_o(S^m)$ at $o = (0, \ldots, 0, 1)$ by ordinary rotations, *on* S^m, χ depends only on the distance $\cos^{-1} x_m$ between $x = (x_0, \ldots, x_m)$ and o. This means that, *on* $\mathbf{R}^{m+1}$, χ is a homogeneous degree p harmonic polynomial depending only on x_m and ρ^2. We write $\chi = \sum_{j=0}^{[p/2]} c_j x_m^{p-2j}\rho^{2j}$, where $c_j \in \mathbf{R}$, $j = 0, \ldots, [p/2]$. The difference $\chi - c_0 H(x_m^p)$ is harmonic and, by (2.1.9), it is a multiple of ρ^2. By (2.1.3), it must be zero. We obtain $\chi = c_0 H(x_m^p)$ as stated.

Returning to the claim, we now show that $SO(m+1)$ acts on $\mathcal{H}^p$ irreducibly. Let $V \subset \mathcal{H}^p$ be an $SO(m+1)$-invariant linear subspace. Let $\alpha \in V^*$ be the linear functional given by $\alpha(\chi) = \chi(o)$, $\chi \in V$. If $V \neq 0$ then, by $SO(m+1)$-invariance, α is not identically zero on V. Thus $\ker \alpha$ is of codimension 1 in V, and since o is left fixed by the subgroup $SO(m)$, it is $SO(m)$-invariant. It follows that $(\ker \alpha)^\perp$ is spanned by a zonal spherical harmonic. We conclude that each (nonzero) invariant subspace of $\mathcal{H}^p$ contains a zonal. On the other hand, we just proved that, up to a con-

stant multiple, $\mathcal{H}^p$ has a unique zonal. Hence $\mathcal{H}^p$ has no proper invariant subspaces. Irreducibility therefore follows.

Restricting $\mathcal{H}^p_m$ to the subgroup $SO(m)$, we next claim that

$$\mathcal{H}^p_m|_{SO(m)} = \sum_{q=0}^{p} \mathcal{H}^q_{m-1} \tag{2.1.15}$$

as $SO(m)$-modules. (We inserted the lower indices to keep track of the domain dimensions.) In this equality we let $\mathcal{H}^q_{m-1}$ denote $H(x_m^{p-q}\mathcal{H}^q_{m-1})$, the image of $x_m^{p-q}\mathcal{H}^q_{m-1} \subset \mathcal{P}^q_{m+1}$ under the harmonic projection $H : \mathcal{P}^p_{m+1} \to \mathcal{H}^p_m$. Since H is $SO(m+1)$-equivariant and $x_m^{p-q}\mathcal{H}^q_{m-1}$ is an irreducible $SO(m)$-module, Schur's lemma implies that H is injective on $x_m^{p-q}\mathcal{H}^q_{m-1}$. Thus, $H(x_m^{p-q}\mathcal{H}^q_{m-1})$ and $\mathcal{H}^q_{m-1}$ are isomorphic as $SO(m)$-modules. We agree that $\mathcal{H}^q_{m-1}$, *when considered as an* $SO(m)$*-submodule of* $\mathcal{H}^p_m$, *will always mean* $H(x_m^{p-q}\mathcal{H}^q_{m-1})$. We obtain that the right-hand side in (2.1.15) is contained in $\mathcal{H}^p_m|_{SO(m)}$. Now (2.1.15) follows from (2.1.5) for reasons of dimension. Indeed, iterating the binomial identity

$$\binom{p+m}{m} = \binom{p+m-1}{m} + \binom{p+m-1}{m-1},$$

we obtain

$$\binom{p+m}{m} = \sum_{q=0}^{p} \binom{q+m-1}{m-1}.$$

Thus, we have

$$\binom{p+m}{m} - \binom{p+m-2}{m} = \sum_{q=0}^{p} \left[\binom{q+m-1}{m-1} - \binom{q+m-3}{m-1} \right],$$

and the differences are exactly the respective dimensions of the modules in (2.1.15). (For a different proof of (2.1.15), see Problem 2.1.)

The decomposition in (2.1.15) just proved is called the *branching rule* for $\mathcal{H}^p_m$ and it is very useful since it allows the use of induction with respect to m. (A more general branching rule for $SO(m) \subset SO(m+1)$ is discussed in Appendix 3.)

Remark. By easy induction in the use of Schur's lemma, it follows that if C is a linear endomorphism of $\mathcal{H}^p_m$ commuting with the action of $SO(m+1)$ then C is a *real* constant multiple of the identity.

There are two basic operators acting on spherical harmonics. These are the directional derivative $\partial_a : \mathcal{H}^p \to \mathcal{H}^{p-1}$ at the direction $a \in \mathbf{R}^{m+1}$, and the operator $\delta_a : \mathcal{H}^p \to \mathcal{H}^{p+1}$, defined by multiplication with the linear functional $a^* \in \mathcal{H}^1 = (\mathbf{R}^{m+1})^*$ (associated to $a \in \mathbf{R}^{m+1}$) followed by harmonic projection:

$$\delta_a \chi = H(a^* \cdot \chi), \quad \chi \in \mathcal{H}^p. \tag{2.1.16}$$

The operators ∂_a and δ_a relate by (2.1.8). We will also prove shortly that, adjusting the degrees, up to constant multiples, they are transposes of each other. For simplicity, we use the notations $\partial_i = \partial_{e_i}$ and $\delta_i = \delta_{e_i}$, where e_i is the i-th vector in the standard basis $\{e_i\}_{i=0}^m \subset \mathbf{R}^{m+1}$.

Since $\mathcal{H}^1 = (\mathbf{R}^{m+1})^*$, varying $a \in \mathbf{R}^{m+1}$, these operators give rise to homomorphisms

$$\iota_\pm : \mathcal{H}^{p\pm1} \to \mathcal{H}^p \otimes \mathcal{H}^1$$

of $SO(m+1)$-modules. We define $\iota_\pm$ up to positive constant factors $c_p^\pm$ (to be determined later) by

$$\iota_-(\chi')(a) = c_p^- \delta_a(\chi') = c_p^- H(a^* \cdot \chi'), \quad \chi' \in \mathcal{H}^{p-1},$$

and

$$\iota_+(\chi'')(a) = c_p^+ \partial_a \chi'', \quad \chi'' \in \mathcal{H}^{p+1}.$$

With respect to the standard basis $\{e_i\}_{i=0}^m \subset \mathbf{R}^{m+1}$, we have

$$\iota_-(\chi') = c_p^- \sum_{i=0}^m \delta_i \chi' \otimes y_i, \ \chi' \in \mathcal{H}^{p-1},$$

and

$$\iota_+(\chi'') = c_p^+ \sum_{i=0}^m \partial_i \chi'' \otimes y_i, \ \chi'' \in \mathcal{H}^{p+1},$$

where we used the variables y_i, $i = 0, \dots, m$, in the second tensor component. Throughout, we use the convention that $\chi \in \mathcal{H}^p$, $\chi' \in \mathcal{H}^{p-1}$ and $\chi'' \in \mathcal{H}^{p+1}$ denote typical elements.

The L^2-scalar product on $\mathcal{P}^p$ restricts to the L^2-scalar product on $\mathcal{H}^p$. For future purposes, however, it will be convenient to scale the latter, and define

$$\langle \chi_1, \chi_2 \rangle = \frac{\dim \mathcal{H}^p}{\operatorname{vol}(S^m)} \langle \chi_1, \chi_2 \rangle_{L^2} = \frac{\dim \mathcal{H}^p}{\operatorname{vol}(S^m)} \int_{S^m} \chi_1 \chi_2 \, v_{S^m}, \ \chi_1, \chi_2 \in \mathcal{H}^p. \tag{2.1.17}$$

Here the volume of the sphere S^m is

$$\operatorname{vol}(S^m) = \frac{2\pi^{\frac{m+1}{2}}}{\Gamma\left(\frac{m+1}{2}\right)},$$

where Γ is the Gamma function. The action of $SO(m+1)$ on the irreducible $\mathcal{H}^p$ leaves this scalar product invariant. We express this by saying that $\mathcal{H}^p$ is an *orthogonal $SO(m+1)$-module.*

The values of the normalizing constants $c_p^\pm$ are determined as follows: By the remark above, the compositions $\iota_\pm^\top \circ \iota_\pm$, being endomorphisms of the irreducible $\mathcal{H}^{p\pm1}$, are constant multiples of the identity. We now require that $\iota_\pm : \mathcal{H}^{p\pm1} \to \mathcal{H}^p \otimes \mathcal{H}^1$ are *linear isometric imbeddings*:

$$\iota_\pm^\top \circ \iota_\pm = I. \tag{2.1.18}$$

These conditions determine $c_p^{\pm}$ uniquely.

The actual values of $c_p^{\pm}$ are given in the following:

Proposition 2.1.3. *For $a \in \mathbf{R}^{m+1}$ and $\chi \in \mathcal{H}^p$, we have*

$$\iota_-^{\top}(\chi \otimes a^*) = \sqrt{\frac{2}{\lambda_{2p}}}\,\partial_a \chi, \tag{2.1.19}$$

$$\iota_+^{\top}(\chi \otimes a^*) = \sqrt{\frac{\lambda_{2p}}{2\lambda_p}}\,\delta_a \chi, \tag{2.1.20}$$

and

$$c_p^- = \frac{p}{\sqrt{\lambda_{2p}/2}} \frac{\lambda_{2(p-1)}}{2\lambda_{p-1}} \tag{2.1.21}$$

$$c_p^+ = \frac{1}{p+1}\sqrt{\frac{2\lambda_p}{\lambda_{2p}}}. \tag{2.1.22}$$

For the proof, we need the fact that, up to constant multiples, the operators ∂_a and δ_a, $a \in \mathbf{R}^{m+1}$, are transposes of each other.

Lemma 2.1.4. *On $\mathcal{H}^p$, we have*

$$\partial_a^{\top} = \mu_p \delta_a, \text{ and } \delta_a^{\top} = \frac{1}{\mu_{p-1}}\partial_a,\ a \in \mathbf{R}^{m+1}, \tag{2.1.23}$$

where

$$\mu_p = (p+1)\frac{\lambda_{2p}}{2\lambda_p}. \tag{2.1.24}$$

Equivalently, for $a \in \mathbf{R}^{m+1}$, $\chi \in \mathcal{H}^p$ and $\chi'' \in \mathcal{H}^{p+1}$, we have

$$\langle \chi, \partial_a \chi'' \rangle = \mu_p \langle H(a^* \cdot \chi), \chi'' \rangle. \tag{2.1.25}$$

PROOF. Up to the constant multiple $\dim \mathcal{H}^p / \operatorname{vol}(S^m)$, the left-hand side of (2.1.25) is

$$\int_{S^m} \chi\, \partial_a \chi''\, v_{S^m}.$$

Spherical harmonics of different order are L^2-orthogonal. Using this and (2.1.8), we compute

$$\begin{aligned}
\int_{S^m} \chi\, \partial_a \chi''\, v_{S^m} &= \int_{S^m} \chi \rho^2 \partial_a \chi''\, v_{S^m} \\
&= \frac{\lambda_{2(p+1)}}{2(p+1)} \int_{S^m} \chi\,(a^* \chi'' - H(a^* \chi''))\, v_{S^m} \\
&= \frac{\lambda_{2(p+1)}}{2(p+1)} \int_{S^m} a^* \chi \chi''\, v_{S^m}
\end{aligned}$$

$$= \frac{\lambda_{2(p+1)}}{2(p+1)} \int_{S^m} H(a^*\chi)\chi'' \, v_{S^m}$$
$$= \frac{\mathrm{vol}\,(S^m)}{\dim \mathcal{H}^{p+1}} \frac{\lambda_{2(p+1)}}{2(p+1)} \langle H(a^*\chi), \chi'' \rangle.$$

Thus, we have

$$\langle \chi, \partial_a \chi'' \rangle = \frac{\dim \mathcal{H}^p}{\dim \mathcal{H}^{p+1}} \frac{\lambda_{2(p+1)}}{2(p+1)} \langle H(a^*\chi), \chi'' \rangle.$$

By (2.1.5), the value of the constant is

$$\frac{\dim \mathcal{H}^p}{\dim \mathcal{H}^{p+1}} \frac{\lambda_{2(p+1)}}{2(p+1)} = \frac{(p+1)(2p+m-1)}{(2p+m+1)(p+m-1)} (2(p+1)+m-1).$$

This is μ_p as defined in (2.1.24). The lemma follows.
For $\xi \in \mathcal{P}^q$, we denote by

$$\partial_\xi : \mathcal{H}^{p+q} \to \mathcal{H}^p,$$

the polynomial differential operator defined by ξ. For example,

$$\triangle = \partial_{\rho^2}.$$

Since ∂_ξ acts on harmonic polynomials, by (2.1.7), we have

$$\partial_\xi = \partial_{H(\xi)}.$$

By induction, we obtain from (2.1.25) the following:

Corollary 2.1.5. *For $\chi_1 \in \mathcal{H}^p$, $\chi_2 \in \mathcal{H}^{p+q}$, and $\xi \in \mathcal{P}^q$, we have*

$$\langle \chi_1, \partial_\xi \chi_2 \rangle = \mu_p \mu_{p+1} \cdots \mu_{p+q-1} \langle H(\xi \cdot \chi_1), \chi_2 \rangle. \tag{2.1.26}$$

Setting $\chi_1 = 1$ in (2.1.26), we have

Corollary 2.1.6. *The scaled L^2-scalar product* (2.1.17) *can be written as*

$$\langle \chi_1, \chi_2 \rangle = \frac{m(m+1)\cdots(m+p-2)}{p!(m+1)(m+3)\cdots(m+2p-3)} \partial_{\chi_1} \chi_2, \quad \chi_1, \chi_2 \in \mathcal{H}^p. \tag{2.1.27}$$

We are now ready to prove Proposition 2.1.3.

Proof of Proposition 2.1.3. Let $\chi \in \mathcal{H}^p$ and $\chi' \in \mathcal{H}^{p-1}$. By (2.1.25), we have

$$\langle \chi \otimes a^*, \iota_-(\chi') \rangle = c_p^- \langle \chi, \delta_a \chi' \rangle = \frac{c_p^-}{\mu_{p-1}} \langle \partial_a \chi, \chi' \rangle$$

so that

$$\iota_-^\top(\chi \otimes a^*) = \frac{c_p^-}{\mu_{p-1}} \partial_a \chi.$$

Using this and (2.1.8), we compute

$$\begin{aligned}
(\iota_-^\top \circ \iota_-)(\chi') &= c_p^- \sum_{i=0}^m \iota_-^\top\Big(\delta_i\chi' \otimes y_i\Big) = \frac{(c_p^-)^2}{\mu_{p-1}} \sum_{i=0}^m \partial_i\delta_i\chi' \\
&= \frac{(c_p^-)^2}{\mu_{p-1}} \sum_{i=0}^m \partial_i\Big(x_i\chi' - \frac{2(p-1)}{\lambda_{2(p-1)}}\rho^2\partial_i\chi'\Big) \\
&= \frac{(c_p^-)^2}{\mu_{p-1}} \sum_{i=0}^m \Big(\chi' + x_i\partial_i\chi' - \frac{4(p-1)}{\lambda_{2(p-1)}}x_i\partial_i\chi'\Big) \\
&= \frac{(c_p^-)^2}{\mu_{p-1}}\Big(m+p-2\frac{p-1}{2p+m-3}\Big)\chi' \\
&= \frac{(c_p^-)^2}{\mu_{p-1}}\frac{(2p+m-1)(p+m-2)}{2p+m-3}\chi' \\
&= \frac{(c_p^-)^2}{\mu_{p-1}}\frac{\lambda_{2p}}{2p}\frac{2\lambda_{p-1}}{\lambda_{2(p-1)}}\chi'.
\end{aligned}$$

(2.1.19) and (2.1.21) follow. The computations for (2.1.20) and (2.1.22) are analogous.

Given $a, b \in \mathbf{R}^{m+1}$, we define the linear map $E_{ab} : \mathcal{H}^p \to \mathcal{H}^p$ by

$$E_{ab} = \delta_b \circ \partial_a - \delta_a \circ \partial_b.$$

By Lemma 2.1.4, E_{ab} is skew on $\mathcal{H}^p$. In fact, we have

$$\begin{aligned}
E_{ab}^\top &= \partial_a^\top \circ \delta_b^\top - \partial_b^\top \circ \delta_a^\top \\
&= \delta_a \circ \partial_b - \delta_b \circ \partial_a \\
&= E_{ba} = -E_{ab}.
\end{aligned}$$

The set $\{E_{ab}\}_{a,b\in\mathbf{R}^{m+1}}$ can be pieced together to a homomorphism

$$\iota_\wedge : \mathcal{H}^p \to \mathcal{H}^p \otimes so(m+1)^*$$

of $SO(m+1)$-modules, by setting

$$\iota_\wedge(\chi)(a \wedge b) = E_{ab}\chi, \quad \chi \in \mathcal{H}^p,\ a, b \in \mathbf{R}^{m+1}.$$

This definition makes sense because $E_{ab} = -E_{ba}$ so that E_{ab} depends only on $a \wedge b \in \wedge^2(\mathbf{R}^{m+1}) = so(m+1)$. In the definition of E_{ab}, harmonic projection can be dispensed with:

$$E_{ab}\chi = b^*\partial_a\chi - a^*\partial_b\chi, \quad \chi \in \mathcal{H}^p,$$

since the right-hand side is harmonic.

Let $a, b \in \mathbf{R}^{m+1}$ be orthogonal unit vectors. Then E_{ab}, acting on $\mathcal{H}^p$, is induced by the infinitesimal rotation in the plane spanned by a and b (Problem 2.5). The operator $\iota_\wedge$ will play an important role in Chapter 4.

2.2 Generalities on Eigenmaps

A map $f : \mathbf{R}^{m+1} \to V$ into a Euclidean vector space V is *spherical* if f maps the unit sphere S^m of $\mathbf{R}^{m+1}$ to the unit sphere S_V of V. A map $f : \mathbf{R}^{m+1} \to V$ is a *p-homogeneous polynomial map* if the components $\alpha \circ f$, $\alpha \in V^*$, of f are homogeneous polynomials of degree p, i.e. $\alpha \circ f \in \mathcal{P}^p$, $\alpha \in V^*$. Given a spherical p-homogeneous polynomial map $f : \mathbf{R}^{m+1} \to V$, its restriction $f : S^m \to S_V$ (denoted by the same symbol) is said to be a *p-form.*[1]

A map $f : \mathbf{R}^{m+1} \to V$ is said to be *harmonic* if the components of f are harmonic functions. If f is a harmonic p-homogeneous polynomial map then, according to the previous section, each component $\alpha \circ f$, $\alpha \in V^*$, is (actually restricts to) a spherical harmonic of order p on S^m. If, in addition, f is spherical then the p-form $f : S^m \to S_V$ (denoted by the same symbol) is called a *p-eigenmap.* According to the previous section, every spherical harmonic of order p on S^m is the restriction of a harmonic p-homogeneous polynomial on $\mathbf{R}^{m+1}$. Hence, if $f : S^m \to S_V$ is a map such that each component $\alpha \circ f$, $\alpha \in V^*$, is a spherical harmonic of order p then f is a p-eigenmap, and f is the restriction (to the respective spheres) of a harmonic p-homogeneous polynomial map $f : \mathbf{R}^{m+1} \to V$.

Example 2.2.1. The equivariant construction of Section 1.4 applied to an $SU(2)$-submodule W of $\mathcal{H}^p|_{SU(2)}$ gives p-eigenmaps $f_\xi : S^3 \to S_W$, $\xi \in S_W$.

A map $f : \mathbf{R}^{m+1} \to V$ (resp. $f : S^m \to S_V$) is *full* if f has no vanishing component, or equivalently, if the image of f is not contained in any proper linear subspace (resp. proper great sphere). Indeed, for $\alpha \in V^*$, $\alpha \circ f$ vanishes iff the image of f is contained in the kernel of α. In general, taking the linear span V_0 of f (resp. the least great sphere S_{V_0} which contains the image of f) defines a full map $f : \mathbf{R}^{m+1} \to V_0$ (resp. $f : S^m \to S_{V_0}$). Clearly, f is p-homogeneous polynomial, harmonic (resp. p-form, p-eigenmap) iff f_0 is.

Two maps $f_1 : \mathbf{R}^{m+1} \to V_1$ and $f_2 : \mathbf{R}^{m+1} \to V_2$ are *congruent*, written as $f_1 \cong f_2$, if there exists a linear isometry $U : V_1 \to V_2$ such that $f_2 = U \circ f_1$. Since linear isometries leave the unit sphere invariant, the same definition applies to maps between spheres. Clearly, if f_1 is p-homogeneous polynomial, harmonic etc. then so is a congruent f_2. Let $f : S^m \to S_V$ be a full p-eigenmap. Precomposition by f defines a linear map $f^* : V^* \to \mathcal{H}^p$ that is injective since f is full. We let $V_f \subset \mathcal{H}^p$ denote the image of this linear imbedding:

$$V_f = \{\alpha \circ f \mid \alpha \in V^*\}.$$

We call V_f the *space of components of f.*

[1] This is not to be confused with the concept of form à la Klein introduced in Section 1.3.

Each spherical harmonic χ in V_f can be written as $\chi = \alpha \circ f$, $\alpha \in V^*$. Since f is full, α is uniquely determined by χ. Restricting to the image, we obtain the linear isomorphism $f^* : V^* \to V_f$. Since V is Euclidean, the "musical isomorphisms" (defined by the scalar product on V) naturally identify V with V^*. With these, we have the isomorphisms

$$V \cong V^* \xrightarrow{f^*} V_f.$$

Under these isomorphisms, the vector $v \in V$, the linear functional $\langle v, \cdot \rangle \in V^*$, and the spherical harmonic $\langle v, f \rangle \in \mathcal{H}^p$ correspond to each other. In particular, for f full, $\dim V = \dim V_f$. Note also that if $f_1 \cong f_2$ then $V_{f_1} = V_{f_2}$.

Let $f : S^m \to S_V$ be a full p-eigenmap. With respect to an orthonormal basis in V, we have $V = \mathbf{R}^{n+1}$, $\dim V = n+1$, and we can write $f : S^m \to S^n$ with *component functions* $f^0, \ldots, f^n$ spanning $V_f \subset \mathcal{H}^p$. We say that f has *orthonormal components* if $f^0, \ldots, f^n$ are orthogonal in $\mathcal{H}^p$ with the same norm with respect to the scalar product (2.1.17) in $\mathcal{H}^p$. The common norm is $\sqrt{\dim \mathcal{H}^p / \dim V}$.

Indeed, integrating $\sum_{j=0}^n (f^j)^2 = 1$ over S^m, we obtain $\sum_{j=0}^n |f^j|^2 = \dim \mathcal{H}^p$, and the claim follows. It is clear that if $f : S^m \to S_V$ has orthonormal components with respect to an orthonormal basis in V, then it has orthonormal components with respect to any orthonormal basis in V. Downplaying the role of the bases, we will simply say that f *has orthonormal components.*

Let $f : S^m \to S_V$ and $f' : S^m \to S_{V'}$ be full p-eigenmaps. f' is said to be *derived* from f, written as $f' \leftharpoonup f$, if $V_{f'} \subset V_f$. Equivalently, every component of f' is also a component of f. Hence $f' \leftharpoonup f$ iff there exists a *surjective* linear map $A : V \to V'$ such that $A \circ f = f'$. If $f' \leftharpoonup f$, then $\dim V_{f'} \leq \dim V_f$ and equality holds iff $f' \rightleftharpoons f$. The relation $\rightharpoonup$ is clearly transitive.

In what follows, for brevity, we set

$$N(p) = N(m,p) = \dim \mathcal{H}^p_m - 1,$$

with the actual value given in (2.1.5). For a fixed orthonormal basis $\{f^j_p = f^j_{m,p}\}_{j=0}^{N(p)} \subset \mathcal{H}^p$, we define the p-homogeneous polynomial map $f_p = f_{m,p} : \mathbf{R}^{m+1} \to \mathcal{H}^p$ by

$$f_p(x) = \sum_{j=0}^{N(p)} f^j_p(x) f^j_p, \quad x \in S^m. \tag{2.2.1}$$

f_p does not depend on the orthonormal basis chosen. Indeed, the transfer matrix between orthonormal bases of $\mathcal{H}^p$ is orthogonal, and the matrix entries cancel in the summation (2.2.1).

Let $\rho_p = \rho_{m,p} : SO(m+1) \to SO(\mathcal{H}^p)$ be the homomorphism that defines the orthogonal $SO(m+1)$-module structure on $\mathcal{H}^p$:

$$\rho_p(g) : \chi \mapsto \chi \circ g^{-1}, \quad \chi \in \mathcal{H}^p,\ g \in SO(m+1).$$

We claim that f_p is *equivariant* with respect to ρ_p:

$$f_p \circ g = \rho_p(g) \circ f_p, \quad g \in SO(m+1). \tag{2.2.2}$$

To prove this, we compute

$$\begin{aligned} f_p(g \cdot x) &= \sum_{j=0}^{N(p)} f_p^j(g \cdot x) f_p^j \\ &= \sum_{j=0}^{N(p)} (\rho_p(g)^{-1} \circ f_p^j)(x) f_p^j \\ &= \sum_{j=0}^{N(p)} f_p^j(x)\, \rho_p(g) \circ f_p^j \\ &= (\rho_p(g) \circ f_p)(x). \end{aligned}$$

Since $SO(m+1)$ acts transitively on S^m, f_p is spherical, and the restriction $f_p : S^m \to S_{\mathcal{H}^p}$ is a p-eigenmap. For $m = 2$ and $p \geq 2$, $f_p = f_{2,p} : S^2 \to S^{2p}$ is (congruent to) the Veronese map Ver_p introduced in Section 1.4. (This follows either from the fact that both have orthonormal components (Problem 2.7), or from a general rigidity result (Corollary 2.5.2). Notice also that we adjusted the notation so that p is the degree of Ver_p.)

For the same reason as for Ver_p in Section 1.4 (transitivity of the isotropy subgroups of $SO(m+1)$ at points of S^m in the unit tangent spheres, and thereby conformality), f_p is a minimal immersion of degree p. We call f_p the *standard minimal immersion of degree p*. Since $V_{f_p} = \mathcal{H}^p$, any full p-eigenmap $f : S^m \to S_V$ is derived from f_p.

Remark. DoCarmo and Wallach [1] were the first to generalize orthonormality of the components of the Veronese map $Ver_d : S^2 \to S^{2d}$ to arrive at the concept of the standard minimal immersion.

Let $f : S^m \to S_V$ be a p-eigenmap and assume that f is *equivariant* with respect to a homomorphism $\rho_f : G \to SO(V)$, where $G \subset SO(m+1)$ is a closed subgroup. Equivariance means that

$$f \circ g = \rho_f(g) \circ f,$$

for all $g \in G$. In particular, the image of f is an orbit in V under the action of G by ρ_f. Since f is full, ρ_f is uniquely determined. As in Chapter 1, we will also say that $f : S^m \to S_V$ is *G-equivariant* without the explicit mention of $\rho_f : G \to SO(V)$. The homomorphism ρ_f defines an orthogonal G-module structure on V (a representation of G on V with the scalar product being G-invariant), and therefore on V^*. Under the isomorphism

$V^* \to V_f \subset \mathcal{H}^p$, V becomes a G-submodule of $\mathcal{H}^p|_G$. Here $g \in G$ acts on V^* as $g \cdot \alpha = \alpha \circ \rho_f(g)^{-1}$, $\alpha \in V^*$, since

$$(g \cdot \alpha) \circ f = g \cdot (\alpha \circ f) = \alpha \circ f \circ g^{-1} = (\alpha \circ \rho_f(g)^{-1}) \circ f.$$

2.3 Moduli

Let V be a Euclidean vector space. The symmetric square S^2V can be viewed as the vector space of symmetric endomorphisms of V. The scalar product on $S^2V \subset \operatorname{Hom}(V,V)$ defined by

$$\langle C, C' \rangle = \operatorname{trace}(C' \cdot C),\ C, C' \in \operatorname{Hom}(V,V)$$

makes S^2V a Euclidean vector space. With respect to an orthonormal basis in V, writing $C = (c_{jj'})_{j,j'=0}^n$ and $C' = (c'_{jj'})_{j,j'=0}^n$, $\dim V = n+1$, we have

$$\langle C, C' \rangle = \sum_{j,j'=0}^{n} c_{jj'} c'_{jj'}.$$

This scalar product corresponds to the usual dot product of vectors when a matrix is considered a vector by "reading off" the entries.

For $v, v' \in V$, let $v \odot v' \in S^2V$ denote the symmetric tensor product of v and v'. By definition,

$$(v \odot v')(w) = \frac{1}{2}(\langle v, w \rangle v' + \langle v', w \rangle v), \quad w \in V.$$

For $v = v' \in S_V$ a unit vector, we have

$$(v \odot v)(w) = \langle v, w \rangle w, \quad w \in V.$$

Geometrically, $v \odot v$ is the orthogonal projection in V to the line $\mathbf{R} \cdot v$.

Since symmetric endomorphisms are diagonalizable:

$$S^2V = \operatorname{span}\{v \odot v \mid v \in S_V\}. \tag{2.3.1}$$

If $C \in S^2V$ and $v, v' \in V$ then, writing C and v, v' in components with respect to an orthonormal basis in V, we have

$$\langle C, v \odot v' \rangle = \sum_{j,j'=0}^{n} c_{jj'} (v \odot v')_{jj'} = \sum_{j,j'=0}^{n} c_{jj'} v_j v'_{j'} = \langle Cv, v' \rangle,$$

where we used symmetry of C.

Let $f : S^m \to S_V$ be a full p-eigenmap. We define

$$\begin{aligned} \mathcal{E}_f &= \{ f(x) \odot f(x) \mid x \in S^m \}^{\perp} \\ &= \{ C \in S^2V \mid \langle Cf(x), f(x) \rangle = 0, x \in S^m \} \subset S^2V, \end{aligned}$$

where $^{\perp}$ denotes orthogonal complement in S^2V. (By the computation above, the defining equality $\langle Cf(x), f(x) \rangle = 0$ is equivalent to $C \perp f(x) \odot f(x)$.)

Finally, let

$$\mathcal{L}_f = \{C \in \mathcal{E}_f \,|\, C + I \geq 0\},$$

where $I = I_V$ is the identity map of V, and "$\geq$" means "positive semidefinite." Looking at the defining relation of $\mathcal{L}_f$, we see that $\mathcal{L}_f$ is a *convex body in* $\mathcal{E}_f$ (a convex set with nonempty interior), and the origin of $\mathcal{E}_f$ is contained in the interior of $\mathcal{L}_f$.

Theorem 2.3.1. *Given a full p-eigenmap $f : S^m \to S_V$, the set of congruence classes of full eigenmaps $f' : S^m \to S_{V'}$ that are derived from f can be parametrized by the convex body $\mathcal{L}_f$. The parametrization is given by associating to the congruence class of f' the endomorphism*

$$\langle f' \rangle_f = A^\top \cdot A - I \in S^2 V, \tag{2.3.2}$$

where $f' = A \circ f$.

Remark. For spherical minimal immersions, Theorem 2.3.1 is due to DoCarmo-Wallach [1], and it is a cornerstone of the classification theory. Its present form, adapted to p-eigenmaps, is taken from Toth [5].

PROOF OF THEOREM 2.3.1. $\langle f' \rangle_f \in S^2 V$ clearly depends only on the congruence class of f'. Letting $x \in S^m$ and using (2.3.2), we compute

$$\begin{aligned} \langle \langle f' \rangle_f, f(x) \odot f(x) \rangle &= \langle (A^\top A - I) f(x), f(x) \rangle \\ &= |Af(x)|^2 - |f(x)|^2 \\ &= |f'(x)|^2 - 1 = 0. \end{aligned}$$

Thus $\langle f' \rangle_f \in \mathcal{E}_f$. Since $A^\top A \geq 0$ for any A, we have $\langle f' \rangle_f \in \mathcal{L}_f$.

To show injectivity of the parametrization, let $f_1 : S^m \to S_{V_1}$ and $f_2 : S^m \to S_{V_2}$ be p-eigenmaps with $f_1 = A_1 \circ f$ and $f_2 = A_2 \circ f$, $A_1 : V \to V_1$ and $A_2 : V \to V_2$ linear and surjective, and assume that $\langle f_1 \rangle_f = \langle f_2 \rangle_f$. Then we have

$$A_1^\top A_1 = A_2^\top A_2. \tag{2.3.3}$$

In particular, A_1 and A_2 have the same kernel $K \subset V$. We may assume that K is zero (or else, restrict A_1 and A_2 to $K^\perp \subset V$). According to the polar decomposition in linear algebra, any linear isomorphism $A_0 : V \to V_0$ between Euclidean vector spaces can be written as $A_0 = U_0 \circ Q_0$, where $U : V \to V_0$ is a linear isometry and $Q_0 \in S^2 V$ is positive definite. Thus we can write $A_1 = U_1 \circ Q_1$ and $A_2 = U_2 \circ Q_2$, where $U_1 : V \to V_1$ and $U_2 : V \to V_2$ are linear isometries and Q_1 and Q_2 are symmetric positive definite endomorphisms of V. Substituting this into (2.3.3) and taking the square root of both sides, we obtain $Q_1 = Q_2$. Thus $A_1 = U_1 \circ U_2^{-1} \circ A_2$, and f_1 and f_2 are congruent. To show surjectivity of the parametrization, let $C \in \mathcal{L}_f$, and define $A = \sqrt{C + I} : V \to V$. Notice that the square root exists since $C + I$ is positive semidefinite. Restricting the range of A to its image, we obtain $A_0 : V \to V_0$, $V_0 = \operatorname{im} A$. Then $f_0 = A_0 \circ f : M \to S_{V_0}$ is

a full eigenmap derived from f with $\langle f_0\rangle_f = A_0^\top A_0 - I = A^2 - I = C$. The theorem follows.

The convex body $\mathcal{L}_f$ of $\mathcal{E}_f(\subset S^2V)$ is called the (relative) *moduli space* associated to the full eigenmap $f : S^m \to S_V$. By (2.3.2), f itself corresponds to the origin: $\langle f\rangle_f = 0$. It also follows that the interior $\operatorname{int}\mathcal{L}_f$ parametrizes those full eigenmaps $f' : S^m \to S_{V'}$ for which $f \rightleftharpoons f'$, or equivalently, $f' = A \circ f$ with $A : V \to V'$ linear and invertible. Since A is always onto, this holds iff $\dim V = \dim V'$. Thus the boundary $\partial\mathcal{L}_f$ corresponds to those eigenmaps $f' : S^m \to S_{V'}$ for which $f' \leftharpoonup f$ and $\dim V' < \dim V$.

Since $\leftharpoonup$ is a transitive relation, for $f' \leftharpoonup f$, $\mathcal{L}_{f'}$ can be imbedded into $\mathcal{L}_f$. We define this imbedding $\iota : \mathcal{L}_{f'} \to \mathcal{L}_f$ by

$$\iota(\langle f''\rangle_{f'}) = \langle f''\rangle_f$$

whenever $f'' \leftharpoonup f' \leftharpoonup f$. We claim that this imbeddding can be extended to an affine map

$$\iota : S^2V' \to S^2V,$$

and the image of $\mathcal{L}_{f'}$ under ι is actually an affine slice of $\mathcal{L}_f$ (the intersection of $\mathcal{L}_f$ with an affine subspace in $\mathcal{E}_f$). To obtain an explicit form of the extended map, we let $f : S^m \to S_V$, $f' : S^m \to S_{V'}$, and $f'' : S^m \to S_{V''}$ be full p-eigenmaps with $f'' \leftharpoonup f' \leftharpoonup f$, i.e. $V_{f''} \subset V_{f'} \subset V_f$. Let $f' = A \circ f$ and $f'' = A' \circ f'$, where $A : V \to V'$ and $A' : V' \to V''$ are surjective linear maps. We have $f'' = (A' \cdot A) \circ f$. By (2.3.2), the points in the moduli corresponding to our eigenmaps are

$$\begin{aligned}\langle f'\rangle_f &= A^\top \cdot A - I_V,\\ \langle f''\rangle_{f'} &= A'^\top \cdot A' - I_{V'},\\ \langle f''\rangle_f &= (A' \cdot A)^\top \cdot (A' \cdot A) - I_V.\end{aligned}$$

By definition

$$\iota(A'^\top \cdot A' - I_{V'}) = (A' \cdot A)^\top \cdot (A' \cdot A) - I_V.$$

Setting $C' = A'^\top \cdot A' - I_{V'} \in S^2V'$, we rewrite this as

$$\iota(C') = A^\top \cdot (C' + I_{V'}) \cdot A - I_V = A^\top \cdot C' \cdot A + \langle f'\rangle_f. \tag{2.3.4}$$

We now extend ι to any $C' \in S^2V'$ by this formula.

Clearly, this extended ι is affine and injective (since A is onto). For $C' \in \mathcal{E}_{f'}$, we have

$$\begin{aligned}\langle \iota(C')f(x), f(x)\rangle &= \langle (C' + I_{V'})f'(x), f'(x)\rangle - |f(x)|^2\\ &= \langle C'f'(x), f'(x)\rangle + |f'(x)|^2 - |f(x)|^2\\ &= \langle C'f'(x), f'(x)\rangle.\end{aligned}$$

This shows that $C' \in \mathcal{E}_{f'}$ iff $\iota(C') \in \mathcal{E}_f$, in particular, ι maps $\mathcal{E}_{f'}$ into $\mathcal{E}_f$. Finally, since $C' + I_{V'} \geq 0$ iff $\iota(C') + I_V = A^\top(C' + I_{V'})A \geq 0$, we see that

$$\iota(\mathcal{L}_{f'}) = \iota(\mathcal{E}_{f'}) \cap \mathcal{L}_f.$$

From now on, we identify $\mathcal{L}_{f'}$ with its image in $\mathcal{L}_f$. Notice that the interior of $\mathcal{L}_{f'} \subset \mathcal{E}_{f'}$ becomes relative interior of $\mathcal{L}_{f'} \subset \mathcal{L}_f$ in $\mathcal{E}_f$. For simplicity, the interior $\operatorname{int} \mathcal{L}_{f'}$ will always mean the relative interior with respect to $\mathcal{E}_{f'}$.

We can also consider the "linearized map"

$$\iota_0 : S^2V' \to S^2V$$

defined by

$$\iota_0(C') = A^\top \cdot C' \cdot A, \quad C' \in S^2V'.$$

We claim that, under the isomorphisms $V \cong V_f$ and $V' \cong V_{f'}$, the image of ι_0 is $S^2(V_{f'})$, where $S^2(V_{f'})$ is considered as a linear subspace of $S^2(V_f)$ via the inclusion $V_{f'} \subset V_f$. Indeed, recall that, under the isomorphism $V \cong V_f$, $v \in V$ corresponds to $\langle v, f\rangle \in V_f$, $C \in S^2V$ corresponds to, say, $C_f \in S^2(V_f)$: $C_f \cdot \langle v, f\rangle = \langle Cv, f\rangle$. Thus, for $C' \in S^2V'$, we have

$$\begin{aligned} \iota_0(C')_f \cdot \langle v, f\rangle &= \langle \iota_0(C')v, f\rangle \\ &= \langle (A^\top C' A)v, f\rangle \\ &= \langle C'(Av), A \circ f\rangle \\ &= \langle C'v', f'\rangle \\ &= C'_{f'} \cdot \langle v', f'\rangle, \end{aligned}$$

where $v' = Av \in V'$. This shows that

$$\operatorname{im} \iota_0 = S^2(V_{f'}). \tag{2.3.5}$$

As before, $C' \in \mathcal{E}_{f'}$ iff $\iota_0(C') \in \mathcal{E}_f$, and ι_0 maps $\mathcal{E}_{f'}$ injectively into $\mathcal{E}_f$. In particular, by (2.3.5), we have the linear isomorphism

$$\mathcal{E}_{f'} \cong S^2(V_{f'}) \cap \mathcal{E}_f. \tag{2.3.6}$$

For $f = f_p$, we denote $\mathcal{L}^p = \mathcal{L}^p_m = \mathcal{L}_{f_p}$ and $\mathcal{E}^p = \mathcal{E}^p_m = \mathcal{E}_{f_p}$. For simplicity, we set $\langle f\rangle_{f_p} = \langle f\rangle$. We call $\mathcal{L}^p$ the *standard moduli space*. Since all p-eigenmaps are derived from f_p, $\mathcal{L}^p$ parametrizes the congruence classes of *all* full p-eigenmaps $f : S^m \to S_V$. By the above, for any full p-eigenmap $f : S^m \to S_V$, $\mathcal{L}_f$ is an affine slice of $\mathcal{L}^p$. Moreover (2.3.6) specializes to

$$\mathcal{E}_f \cong S^2(V_f) \cap \mathcal{E}^p. \tag{2.3.7}$$

We say that a full p-eigenmap $f : S^m \to S_V$ is of *boundary type* if $\dim V < \dim \mathcal{H}^p$, or equivalently, if $\langle f\rangle \in \partial\mathcal{L}^p$.

Remark. (2.3.7) was first noticed by Weingart; for a special case, see Escher-Weingart [1]. The proof here is new.

Lemma 2.3.2. *Let $f : S^m \to S_V$ be a full p-eigenmap and assume that f has orthonormal components. Then $\mathcal{E}_f \subset S^2V$ consists of traceless symmetric endomorphisms.*

PROOF. By definition, $C \in \mathcal{E}_f$, iff

$$\langle Cf(x), f(x)\rangle = 0,$$

for all $x \in S^m$. In coordinates,

$$\sum_{j,j'=0}^{n} c_{jj'} f^j(x) f^{j'}(x) = 0,$$

where $\dim V = n+1$. Integrating, and using the fact that the components f^j are orthogonal with the same norm, we obtain

$$\sum_{j=0}^{n} c_{jj} = \operatorname{trace} C = 0.$$

Corollary 2.3.3. *$\mathcal{L}_f$ is compact.*

PROOF. It is enough to prove that $\mathcal{L}^p$ is compact. By Lemma 2.3.2, $\mathcal{E}^p$ consists of traceless symmetric endomorphisms. On the other hand, if $C \in \mathcal{L}^p$ then $C + I \geq 0$ so that the eigenvalues of C are ≥ -1. Since the trace of C is zero, the eigenvalues are also bounded from above. Compactness therefore follows.

Corollary 2.3.4. *$SO(m+1)$ acts on $\mathcal{E}^p$ with no nonzero fixed points.*

PROOF. If $C \in \mathcal{E}^p$ is $SO(m+1)$-fixed, then, as a symmetric endomorphism of $\mathcal{H}^p$, C commutes with the $SO(m+1)$-module structure on $\mathcal{H}^p$. By Section 2.1, C is a real constant multiple of the identity. Since C is traceless (Lemma 2.3.2) it must be zero.

We will encounter situations where the argument in the proof of Theorem 2.3.1 goes through with appropriate modifications depending on the geometric objects to be parametrized. In these cases we will say that a *DoCarmo-Wallach type argument applies.* As an example, we see that a DoCarmo-Wallach type argument applies to the set of congruence classes of full harmonic p-homogeneous polynomial maps $f : \mathbf{R}^{m+1} \to V$ so that this set can be parametrized by the convex (noncompact) body $\mathcal{K}^p$ of $S^2(\mathcal{H}^p)$ given by

$$\mathcal{K}^p = \{C \in S^2(\mathcal{H}^p) \mid C + I \geq 0\}.$$

As usual, the parametrization is given by

$$f \mapsto \langle f \rangle = C = A^\top A - I,$$

where $f = A \circ f_p$ with $A : \mathcal{H}^p \to V$ is linear and surjective.

Adding the condition of sphericality, we obtain

$$\mathcal{L}^p = \mathcal{K}^p \cap \mathcal{E}^p,$$

i.e. the standard moduli space $\mathcal{L}^p$ is a linear slice of $\mathcal{K}^p$ cut out by $\mathcal{E}^p$. (For another example, see Problem 2.10.)

Theorem 2.3.5. *Let $f_1 : S^m \to S_{V_1}$ and $f_2 : S^m \to S_{V_2}$ be incongruent full p-eigenmaps. Let $\lambda_1, \lambda_2 > 0$ with $\lambda_1 + \lambda_2 = 1$. Then the point*

$$\lambda_1\langle f_1\rangle + \lambda_2\langle f_2\rangle \in \mathcal{L}^p$$

on the segment connecting $\langle f_1\rangle$ and $\langle f_2\rangle$ can be represented by the p-eigenmap $f : S^m \to S_V$, $V = V_1 \times V_2$, given by $f = (\sqrt{\lambda_1} f_1, \sqrt{\lambda_2} f_2)$ made full. In particular, $f \multimap f_1, f_2$ and

$$V_f = V_{f_1} + V_{f_2}$$

so that

$$\dim V_f = \dim V_{f_1} + \dim V_{f_2} - \dim(V_{f_1} \cap V_{f_2}).$$

PROOF. Let $f_1 = A_1 \circ f_p$ and $f_2 = A_2 \circ f_p$ with $A_1 : \mathcal{H}^p \to V_1$ and $A_2 : \mathcal{H}^p \to V_2$ linear and surjective. By (2.3.2), we have

$$\langle f_1\rangle = A_1^\top A_1 - I \quad \text{and} \quad \langle f_2\rangle = A_2^\top A_2 - I.$$

By definition, $f = (\sqrt{\lambda_1} f_1, \sqrt{\lambda_2} f_2) = (\sqrt{\lambda_1} A_1, \sqrt{\lambda_2} A_2) \circ f_p$. Thus

$$\begin{aligned}\langle f\rangle &= (\lambda_1 A_1^\top A_1 + \lambda_2 A_2^\top A_2) - I \\ &= \lambda_1(A_1^\top A_1 - I) + \lambda_2(A_2^\top A_2 - I) \\ &= \lambda_1\langle f_1\rangle + \lambda_2\langle f_2\rangle,\end{aligned}$$

where we used $\lambda_1 + \lambda_2 = 1$. The rest is clear.

Given a convex set $\mathcal{L}$ in a finite dimensional vector space $\mathcal{E}$, we say that a point $C \in \mathcal{L}$ is *extremal* if C is not in the interior of any (nontrivial) segment contained in $\mathcal{L}$. Clearly, C is extremal iff $\mathcal{L} - \{C\}$ is still convex.

A full p-eigenmap $f : S^m \to S_V$ is said to be *linearly rigid* if $\mathcal{L}_f$ is trivial, i.e. if $\mathcal{L}_f$ consists of $\langle f\rangle_f$ alone. Equivalently, f is linearly rigid if $f' \leftharpoonup f$ implies that f' is congruent to f. For example, if a p-eigenmap $f : S^m \to S_V$ is surjective then it is linearly rigid. Indeed, if f is surjective then we have

$$\begin{aligned}\mathcal{E}_f &= \{\, f(x) \odot f(x) \,|\, x \in S^m \}^\perp \\ &= \{\, v \odot v \,|\, v \in S_V \}^\perp \\ &= (S^2 V)^\perp = \{0\},\end{aligned}$$

where we used (2.3.1).

Lemma 2.3.6. *Let $f : S^m \to S_V$ be a full p-eigenmap. Then the extremal points of the moduli space $\mathcal{L}_f$ parametrize the linearly rigid full p-eigenmaps derived from f.*

PROOF. Let $f' : S^m \to S_V$ be a full p-eigenmap with $f' \leftharpoonup f$ and assume that $\langle f'\rangle_f$ is not extremal. Let $\langle f'\rangle_f$ be contained in a segment with endpoints $\langle f_1\rangle_f$ and $\langle f_2\rangle_f$ in $\mathcal{L}_f$. By Lemma 2.3.5, we have

$V_{f'} = V_{f_1} + V_{f_2}$. In particular, $V_{f_1}, V_{f_2} \subset V_{f'}$ so that $f_1, f_2 \leftarrow f'$. We thus have $\langle f_1 \rangle_f, \langle f_2 \rangle_f \in \mathcal{L}_{f'}$ and $\mathcal{L}_{f'}$ is nontrivial.

For the converse statement, let $f' : S^m \to S_V$ be a full linearly nonrigid p-eigenmap with $f' \leftarrow f$. Then $\mathcal{L}_{f'}$ is nontrivial. Since $\langle f' \rangle_f$ is in the interior of $\mathcal{L}_{f'}$, there is a nontrivial segment in $\mathcal{L}_{f'} \subset \mathcal{L}_f$ passing through $\langle f' \rangle_f$, so that this point cannot be extremal.

The Krein-Milman theorem (Theorem A.1.1 in Appendix 1) asserts that a compact convex set in a finite dimensional vector space is the convex hull of its extremal points. We thus have the following:

Theorem 2.3.7. *Given a full p-eigenmap $f : S^m \to S_V$, the moduli space $\mathcal{L}_f$ is the convex hull of the points that correspond to the linearly rigid full p-eigenmaps derived from f. Moreover, this set of extremal points is minimal in the sense that the convex hull of any proper subset of extremal points is a proper subset of $\mathcal{L}_f$.*

Remark. The concept of linear rigidity was introduced by Wallach [1] (Definition 10.1, p.29). The connection between linear rigidity of eigenmaps and extremal points of the moduli is contained in Toth-Ziller [1].

Let $f : S^m \to S_V$ be a p-eigenmap and assume that f is equivariant with respect to a homomorphism $\rho_f : G \to SO(V)$, where $G \subset SO(m+1)$ is a closed subgroup. Recall that, under the isomorphism $V \cong V^* \to V_f \subset \mathcal{H}^p$, V becomes a G-submodule of $\mathcal{H}^p|_G$. The G-module structure on V extends to that of S^2V, and it is given by

$$g \cdot C = \rho_f(g) \cdot C \cdot \rho_f(g)^{-1}, \quad C \in S^2V, \ g \in G.$$

$\mathcal{L}_f$ is G-invariant. More precisely, for $f' : S^m \to S_{V'}$ a full p-eigenmap with $f' \leftarrow f$, we have

$$g \cdot \langle f' \rangle = \langle f' \circ g^{-1} \rangle. \tag{2.3.8}$$

Indeed, let $f' = A \circ f$, where $A : V \to V'$ is linear and surjective. For $g \in G$, equivariance of f implies that

$$\begin{aligned} f' \circ g^{-1} &= A \circ f \circ g^{-1} \\ &= A \circ \rho_f(g^{-1}) \circ f \\ &= A \circ \rho_f(g)^{-1} \circ f. \end{aligned}$$

Taking the corresponding point in $\mathcal{L}_f$, we have

$$\begin{aligned} g \cdot \langle f' \rangle_f &= g \cdot (A^\top A - I) = \rho_f(g)(A^\top A - I)\rho_f(g)^{-1} \\ &= (A \cdot \rho_f(g)^{-1})^\top (A \cdot \rho_f(g)^{-1}) - I = \langle f' \circ g^{-1} \rangle. \end{aligned}$$

(2.3.8) follows. The isotropy group $G_{\langle f' \rangle_f}$ at $\langle f' \rangle_f$ is

$$G_{\langle f' \rangle_f} = \{ g \in G \,|\, U \circ f' = f' \circ g, \text{ for some } U \in SO(V') \}.$$

This is because $\mathcal{L}_f$ parametrizes the congruence classes. In particular, given a (closed) subgroup $G' \subset G$, the fixed point set $(\mathcal{L}_f)^{G'}$ parametrizes the

congruence classes of full G'-equivariant p-eigenmaps $f' : S^m \to S_V$ that are derived from f.

As an application, we see that, by Corollary 2.3.4, *a full $SO(m+1)$-equivariant p-eigenmap must be standard.*

The relation $\rightleftharpoons$ on $\mathcal{L}_f$ is an equivalence and the equivalence classes are the (relative) interiors $\operatorname{int} \mathcal{L}_{f'}$, $f' \leftharpoonup f$, of the relative moduli in $\mathcal{L}_f$.

Remark. The equivalence $\rightleftharpoons$ defines a *cell division* of $\mathcal{L}_f$ in the sense that each cell (equivalence class) is homeomorphic with an open ball and the homeomorphism can be extended to a continuous map to the boundary. In addition, the boundary of a cell is the union of cells of lesser dimension (see Thurston [1], pp. 18-19). This cell division is, in general, not differentiable.

If, in addition, f is equivariant with respect to a homomorphism $\rho : G \to SO(V)$, where $G \subset SO(m+1)$ is a closed subgroup, then G acts on $\mathcal{L}_f$ via (2.3.8). This action preserves $\rightharpoonup$ (since $(f \rightharpoonup) f' \rightharpoonup f''$ implies $f' \circ g \rightharpoonup f'' \circ g$, $g \in G$), and hence it permutes the equivalence classes. The following result shows that G acts "transversally" on these equivalence classes.

Theorem 2.3.8. *Let $f : S^m \to S_V$ be a full G-equivariant p-eigenmap. Let $f' : S^m \to S_{V'}$ be a full linearly nonrigid p-eigenmap derived from f. If $g : \mathbf{R} \to G$ is a 1-parameter subgroup such that the orbit $t \mapsto g(t) \cdot \langle f' \rangle_f$, $t \in \mathbf{R}$, is tangent to $\operatorname{int} \mathcal{L}_{f'}$ at $t = 0$ then this orbit is entirely contained in $\operatorname{int} \mathcal{L}_{f'}$.*

PROOF. As usual, let $f' = A \circ f$ with $A : V \to V'$ linear and surjective. We set

$$U(t) = \rho_f(g(t)) = e^{tB} \in SO(V), \quad t \in \mathbf{R},$$

with $B = (d/dt)U(t)|_{t=0} \in so(V)$. We have

$$g(t) \cdot \langle f' \rangle_f = \langle f' \circ g(t)^{-1} \rangle = U(t) A^\top A U(t)^\top - I.$$

Differentiating, we obtain

$$\frac{d}{dt}(g(t) \cdot \langle f' \rangle_f)_{t=0} = BA^\top A - A^\top AB.$$

Using the explicit form (2.3.4) of the imbedding $\iota : \mathcal{L}_{f'} \to \mathcal{L}_f$, the condition of tangency means that there exists $C' \in \mathcal{E}_{f'}$ such that

$$BA^\top A - A^\top AB = A^\top C' A.$$

We now claim that this implies the existence of a linear map $B' : V' \to V'$ such that

$$AB = B'A. \tag{2.3.9}$$

In fact, since A is onto, $A^\top$ is injective so that B leaves $K = \ker A$ invariant. But B is skew and hence it also leaves $K^\perp$ invariant. We now define

$$B' = (AB)|_K \circ (A|_{K^\perp})^{-1}.$$

Exponentiating the commutation relation (2.3.9), we obtain

$$Ae^{-tB} = AU(t)^{\top} = e^{-tB'}A.$$

Precomposing both sides of this with f, we get

$$f' \circ g(t)^{-1} = e^{-tB'} \circ f'.$$

In terms of the moduli space, this means that

$$g(t) \cdot \langle f' \rangle_f \in \operatorname{int} \mathcal{L}_{f'}.$$

The theorem follows.

Corollary 2.3.9. *Let $f : S^m \to S_V$ be a full G-equivariant p-eigenmap. Let $f' : S^m \to S_{V'}$ be derived from f such that $\langle f' \rangle_f$ is the centroid of $\mathcal{L}_{f'} \subset \mathcal{L}_f$. Then the G-orbit $G(\langle f' \rangle_f)$ intersects $\operatorname{int} \mathcal{L}_{f'}$ only at $\langle f' \rangle_f$. Moreover, $G(\operatorname{int} \mathcal{L}_{f'})$ is a smooth submanifold of $\mathcal{E}_f$. In fact, $G(\operatorname{int} \mathcal{L}_{f'})$ is the total space of a fibre bundle over the G-orbit $G(\langle f' \rangle_f)$, where the projection*

$$\pi : G(\operatorname{int} \mathcal{L}_{f'}) \to G(\langle f' \rangle_f)$$

is obtained by associating to a point the centroid of the interior of the relative moduli in which the point is contained.

PROOF. Let $g \in G$ be such that $g \cdot \langle f' \rangle_f \in \mathcal{L}_{f'}$. Since g maps interiors of relative moduli into one another, it must map $\operatorname{int} \mathcal{L}_{f'}$ onto itself. The centroid of $\operatorname{int} \mathcal{L}_{f'}$ then must be fixed by g. By assumption, this centroid is $\langle f' \rangle_f$. We obtain $g \cdot \langle f' \rangle_f = \langle f' \rangle_f$ and the first statement follows. For any $g \in G$, the centroid of $g \cdot \mathcal{L}_{f'}$ is $g \cdot \langle f' \rangle_f$. Hence the projection π above is well-defined. Smoothness, and in fact local triviality of π follows by selecting a smooth local cross section of the orbit map $G \to G(\langle f' \rangle_f)$.

In the rest of this section we will take a closer look at the DoCarmo-Wallach parametrization (2.3.2), and derive various facts about the geometry of $\mathcal{L}^p$. In most cases this parametrization is inadequate for technical purposes since it requires the knowledge of the components of the standard minimal immersion f_p. Although an explicit orthonormal basis in $\mathcal{H}^p$ can be written down in terms of ultraspherical polynomials (Vilenkin [1]), the actual formulas are complicated and difficult to work with. It is thus desirable to look for other ways to express the parameter point $\langle f \rangle \in \mathcal{L}^p$ corresponding to a full p-eigenmap $f : S^m \to S_V$, especially to make $\langle f \rangle$ computable in terms of the components of f.

Remark. Weingart [1] describes another method for obtaining a parametrization of the moduli without the use of the standard minimal immersion. His approach (adapted here for eigenmaps) is as follows:

Given a harmonic p-homogeneous polynomial map $f : \mathbf{R}^{m+1} \to V$, precomposition by the restriction $f|_{S^m}$ defines a linear map $f^* : V^* \to \mathcal{H}^p$, $f^*(\alpha) = \alpha \circ f$, $\alpha \in V^*$. (To simplify the notation, we suppress the restriction. This conforms to our earlier convention not to distinguish between a

harmonic homogeneous polynomial on $\mathbf{R}^{m+1}$ and its restriction to S^m as a spherical harmonic.) With respect to an orthonormal basis in $V \cong \mathbf{R}^{n+1}$, under f^*, the elements of the dual basis in V^* correspond to the components f^j, $j = 0, \ldots, n$, of f. The symmetric square of f^* gives a linear map $S^2(f^*) : S^2(V^*) \to S^2(\mathcal{H}^p)$. We call the image of the Euclidean scalar product $\langle ., . \rangle_V$ under $S^2(f^*)$ the *eigenform* G_f of f:

$$G_f = S^2(f^*)(\langle ., . \rangle_V).$$

Once again, with respect to an orthonormal basis in $V \cong \mathbf{R}^{n+1}$, we have

$$G_f = \sum_{j=0}^{n} f^j \odot f^j.$$

It is clear that G_f is positive semidefinite. It is equally clear that G_f depends only on the congruence class of f. We consider the composition

$$S^m \xrightarrow{\delta} (\mathcal{H}^p)^* \xrightarrow{(f^*)^*} V,$$

where δ is the Dirac delta functional[2] (defined by evaluating spherical harmonics at the points of S^m), and $(f^*)^*$ is the dual of f^*. We claim that this composition is f. To show this, for $\alpha \in V^*$ and $x \in S^m$, we compute

$$\alpha[(f^*)^* \delta_x] = f^*(\alpha)[\delta_x] = (\alpha \circ f)[\delta_x] = \alpha(f(x)),$$

and the claim follows. The dual map $(f^*)^*$ can be factored through the kernel of G_f, and it gives the sequence

$$S^m \xrightarrow{\delta} (\mathcal{H}^p)^* \xrightarrow{\pi} (\mathcal{H}^p)^* / \ker G_f \xrightarrow{(f^*)^*} V,$$

where, by the very definition of G_f, the last map is an isometry with respect to the scalar product on $(\mathcal{H}^p)^* / \ker G_f$ induced by G_f. This shows that G_f determines f up to congruence. Finally, for $x \in S^m$, we have

$$|f(x)|^2 = \langle (f^*)^* \delta_x, (f^*)^* \delta_x \rangle = S^2(f^*)(\langle \delta_x, \delta_x \rangle) = G_f(\delta_x, \delta_x).$$

Conversely, given a positive semidefinite bilinear form $G \in S^2(\mathcal{H}^p)$, we can form the composition

$$S^m \xrightarrow{\delta} (\mathcal{H}^p)^* \xrightarrow{\pi} (\mathcal{H}^p)^* / \ker G,$$

and ask whether $G = G_f$ for a p-eigenmap $f : S^m \to S_V$, where $V = (\mathcal{H}^p)^* / \ker G$ is endowed with the scalar product induced by G. The answer is clear: A positive semidefinite symmetric bilinear form $G \in S^2(\mathcal{H}^p)$ is the eigenform of a p-eigenmap iff

$$G(\delta_x, \delta_x) = 1, \quad x \in S^m,$$

[2]This is not to be confused with δ_a, $a \in \mathbf{R}^{m+1}$, defined in Section 2.1.

and, in this case, the p-eigenmap whose eigenform is G is the composition above. We conclude that the moduli $\mathcal{L}^p$ can be parametrized by positive semidefinite bilinear forms $G \in S^2(\mathcal{H}^p)$ satisfying this condition.

In this setting, the Dirac delta functional

$$\delta : S^m \to S_{(\mathcal{H}^p)^*}$$

can be considered as the standard minimal immersion of degree p. (δ is $SO(m+1)$-equivariant since, for $x \in S^m$, we have

$$(\delta \circ g)(x) = \delta_{g(x)} = \delta_x \circ g^{-1} = g \cdot \delta_x.$$

Being a p-eigenmap, δ must be standard (Corollary 2.3.4).)

We let $f = A \circ f_p$, where $A : \mathcal{H}^p \to V$ is linear and surjective. An orthonormal basis in $V \cong \mathbf{R}^{n+1}$ allows us to view A as a matrix with entries a_{jl}, $j = 0, \ldots, n$, $l = 0, \ldots, N(p)$. With this, we compute

$$\begin{aligned} G_f = \sum_{j=0}^{n} f^j \odot f^j &= \sum_{j=0}^{n} \sum_{l,l'=0}^{N(p)} a_{jl} a_{jl'} f_p^l \odot f_p^{l'} \\ &= \sum_{l,l'=0}^{N(p)} (A^\top A)_{ll'} f_p^l \odot f_p^{l'} \\ &= \sum_{l,l'=0}^{N(p)} (\langle f \rangle + I)_{ll'} f_p^l \odot f_p^{l'}. \end{aligned}$$

Hence, with the proper identifications, we have

$$G_f = \langle f \rangle + I.$$

We now return to the original setting, and describe our new parametrization of the moduli. Again for technical convenience we will fix an orthonormal basis in V and write our full p-eigenmap as $f : S^m \to S^n$, $V \cong \mathbf{R}^{n+1}$. With this $A : V \to \mathcal{H}^p$ in $f = A \circ f_p$ becomes an $(n+1) \times (N(p)+1)$-matrix. We define the *Gram matrix* $G(f)$ of f as the $(n+1) \times (n+1)$-matrix whose jj'-entry is $\langle f^j, f^{j'} \rangle$, where f^j, $j = 0, \ldots, n$, are the components of f and the scalar product is given by (2.1.17). (The Gram matrix $G(f)$ depends on the choice of the orthonormal basis in V. However, different orthonormal bases give conjugate Gram matrices (Problem 2.11).) Since $f : S^m \to S^n$ is full, the Gram matrix is positive definite.

Example 2.3.10. Writing the Hopf map from Example 1.4.2 in terms of real variables $(x, y, u, v) \in S^3 \subset \mathbf{R}^4$, $z = x + iy$, $w = u + iv$, we have

$$Hopf\,(x, y, u, v) = (x^2 + y^2 - u^2 - v^2, 2(xu + yv), 2(yu - xv)).$$

Using the scalar product (2.1.27), a quick computation shows us that the Gram matrix is $G(Hopf) = 3 \cdot I_3$.

Example 2.3.11. In a similar vein, the complex Veronese map in Example 1.4.1, written in terms of real variables $(x, y, u, v) \in S^3$ takes the form

$$Ver^{\mathbf{C}}(x, y, u, v) = (x^2 - y^2, 2xy, \sqrt{2}(xu - yv), \sqrt{2}(yu + xv), u^2 - v^2, 2uv).$$

The Gram matrix is $G(Ver^{\mathbf{C}}) = 3/2 \cdot I_6$.

Example 2.3.12. In general, the p-eigenmaps $f_\xi : S^3 \to S_{W_p} = S^{2p+1}$ for p odd, and $f_\xi : S^3 \to S_{R_p} = S^p$ for p even, have orthonormal components (Problem 1.29). The common value of the norm square is $\dim \mathcal{H}_3^p / \dim W_p = (p+1)/2$ for p odd, and $\dim \mathcal{H}_3^p / \dim R_p = p + 1$ for p even. It follows that the Gram matrix

$$G(f_\xi) = \begin{cases} \frac{p+1}{2} I_{p+1}, & \text{if } p \text{ is odd,} \\ (p+1) I_{p+1}, & \text{if } p \text{ is even.} \end{cases}$$

For example, for the tetrahedral $Tet : S^3 \to S^6$, octahedral $Oct : S^3 \to S^8$, and icosahedral $Ico : S^3 \to S^{12}$ minimal immersions (Section 1.5), we have

$$G(Tet) = 7I_7, \quad G(Oct) = 9I_9, \quad G(Ico) = 13I_{13}.$$

Let $f : S^m \to S_V$ be a full p-eigenmap with $f = A \circ f_p$, $A : \mathcal{H}^p \to V$ linear and surjective. The Gram matrix $G(f)$ can be expressed in terms of A as follows

$$G(f) = A \cdot A^\top. \tag{2.3.10}$$

To prove this we write f in coordinates, $f^j = \sum_{l=0}^{N(p)} a_{jl} f_p^l$. We have

$$G(f)_{jj'} = \langle f^j, f^{j'} \rangle = \sum_{l,l'=0}^{N(p)} a_{jl} a_{j'l'} \langle f_p^l, f_p^{l'} \rangle = \sum_{l=0}^{N(p)} a_{jl} a_{j'l} = (AA^\top)_{jj'}.$$

(2.3.10) follows. By definition, V_f is spanned by the components f^j, $j = 0, \ldots, n$. The significance of the Gram matrix is apparent from the fact that

$$\left\{ \sum_{j'=0}^{n} G(f)_{jj'}^{-1/2} f^{j'} \right\}_{j=0}^{n} \subset V_f \tag{2.3.11}$$

is an *orthonormal* basis in V_f.

Proposition 2.3.13. *Let $f : S^m \to S^n$ be a full p-eigenmap, and $C = \langle f \rangle \in S^2(\mathcal{H}^p)$. Then $C + I : \mathcal{H}^p \to \mathcal{H}^p$ has image V_f, it leaves V_f invariant, and the restriction $C + I \,|\, V_f$ is positive definite. With respect to the orthonormal basis (2.3.11), we have*

$$C + I \,|\, V_f = G(f). \tag{2.3.12}$$

In particular, the eigenvalues of $G(f)$ are the same as the nonzero eigenvalues of $C + I$ counted with multiplicity.

PROOF. To work out $(C+I)f^j$, $j=0,\dots,n$, we first note that, by (2.3.2) and (2.3.10), we have

$$A(C+I) = A(A^\top A) = G(f)A.$$

Using this we compute

$$\begin{aligned}(C+I)f^j &= \sum_{l=0}^{N(p)} a_{jl}(C+I)f_p^l = \sum_{l,l'=0}^{N(p)} a_{jl}(C+I)_{ll'}f_p^{l'} \\ &= \sum_{l'=0}^{N(p)} (A(C+I))_{jl'}f_p^{l'} = \sum_{l'=0}^{N(p)} (G(f)A)_{jl'}f_p^{l'} \\ &= \sum_{j'=0}^{n}\sum_{l'=0}^{N(p)} G(f)_{jj'}a_{j'l'}f_p^{l'} = \sum_{j'=0}^{n} G(f)_{jj'}f^{j'}.\end{aligned}$$

We obtain that $C+I$ leaves V_f invariant. (This also shows that, under the isomorphism $V \cong V_f$, the restriction of $C+I$ to V_f is simply $G(f)$.) Making the scalar product with another component of f, we have

$$\langle (C+I)f^j, f^{j'}\rangle = G(f)^2_{jj'}.$$

In particular, $C+I$ is positive definite on V_f. Since $\operatorname{rank}(C+I) = \dim V = \dim V_f$, it follows that the image of $C+I$ is V_f. Finally, to show (2.3.12), we compute

$$\begin{aligned}&\left\langle (C+I)\sum_{j'=0}^{n} G(f)^{-1/2}_{jj'}f^{j'}, \sum_{k'=0}^{n} G(f)^{-1/2}_{kk'}f^{k'}\right\rangle \\ &\quad = (G(f)^{-1/2}G(f)^2G(f)^{-1/2})_{jk} = G(f)_{jk}.\end{aligned}$$

The proposition follows.

Assume that $\mathcal{L}^p$ is nontrivial. (This happens iff $m \geq 3$ and $p \geq 2$, see Section 2.5.) Recall that the origin is an interior point of $\mathcal{L}^p$. By convexity and compactness of $\mathcal{L}^p$ (Corollary 2.3.3), any line in $\mathcal{E}^p$ through the origin intersects $\partial\mathcal{L}^p$ in exactly two points. (This follows from elementary convex geometry, see Berger [1], Chapter 11.) Thus, given a full p-eigenmap $f : S^m \to S_V$ of boundary type, $\langle f\rangle \in \partial\mathcal{L}^p$, the line $\mathbf{R}\cdot\langle f\rangle$ intersects $\partial\mathcal{L}^p$ at $\langle f\rangle$ and at another point called the *antipodal* of $\langle f\rangle$. A representative $f^o : S^m \to S_{V^o}$ of the antipodal of $\langle f\rangle$ is called an *antipodal p-eigenmap* of f. f^o is unique up to congruence.

As shown in Appendix 1 the *distortion* of $\mathcal{L}^p$ at $C \in \partial\mathcal{L}^p$ is defined as

$$\lambda_0(C) = \frac{|C|}{|C^o|}, \tag{2.3.13}$$

where C^o is the antipodal of C. The next result implies that the smaller the range dimension is for a full p-eigenmap f, the larger the distortion of $\mathcal{L}^p$ is at $\langle f\rangle$.

Theorem 2.3.14. *Let $f : S^m \to S_V$ be a full p-eigenmap of boundary type: $C = \langle f \rangle \in \partial\mathcal{L}^p$. Then the distortion $\lambda_0(C)$ of $\mathcal{L}^p$ is the maximal eigenvalue of C as a symmetric endomorphism of $\mathcal{H}^p$. The multiplicity $\nu(C)$ of this eigenvalue satisfies $\nu(C) \leq \dim V$, and we have*

$$\frac{\dim \mathcal{H}^p}{\dim V} \leq \lambda_0(C) + 1 \leq \frac{\dim \mathcal{H}^p}{\nu(C)}. \tag{2.3.14}$$

For the antipodal C^o of C, the maximal eigenvalue is $\lambda_0(C^o) = 1/\lambda_0(C)$, its multiplicity is $\nu(C^o) = \dim \mathcal{H}^p - \dim V$, and the range dimension for the p-eigenmap $f^o : S^m \to S_{V^o}$ antipodal to f is $\dim V^o = \dim \mathcal{H}^p - \nu(C)$. $f : S^m \to S_V$ has orthonormal components iff $\nu(C) = \dim V$; in particular, in this case equalities hold in (2.3.14).

PROOF. Consider $-tC + I$ for $t > 0$. Since C is traceless (Lemma 2.3.2) the largest eigenvalue $\lambda_0(C)$ of C is positive, and thus there is a maximal interval $[0, t_0)$ such that $-tC + I$ is positive definite for $0 \leq t < t_0$. By the definition of the moduli $\mathcal{L}^p$, for $0 \leq t < t_0$, $-tC$ is in the interior of $\mathcal{L}^p$. At t_0, the determinant of $-t_0C + I$ vanishes, and so $-t_0C = C^o \in \partial\mathcal{L}^p$. Since $\lambda_0(C)$ is the largest eigenvalue of C, we have $-t_0\lambda_0(C) + 1 = 0$, or equivalently, $t_0 = 1/\lambda_0(C)$. We obtain

$$C^o = -\frac{1}{\lambda_0(C)} C. \tag{2.3.15}$$

Taking norms, we arrive at

$$\lambda_0(C) = \frac{|C|}{|C^o|}. \tag{2.3.16}$$

The first statement follows. (Notice the formal coincidence of this with (2.3.13), and the different meanings of λ_0.)

For simplicity, we let $\lambda_0 = \lambda_0(C)$, $\nu = \nu(C)$, and $n + 1 = \dim V$. Since $C + I$ is positive semidefinite, all eigenvalues of C are ≥ -1. Since rank $(C + I) = \dim V = n + 1$, the multiplicity of the eigenvalue -1 is $N(p) - n$. Let λ_j, $j = 0, \ldots, n$, be the eigenvalues > -1 of C in decreasing order. Since the largest eigenvalue λ_0 has multiplicity ν, we have $\nu \leq n+1$. By (2.3.15), the largest eigenvalue of C^o is $1/\lambda_0$ with multiplicity $N(p)-n$, and the smallest eigenvalue -1 is of multiplicity ν. In particular, rank $(C^o+I) = N(p)+1-\nu$. To obtain (2.3.14), we first write the condition that C is traceless in the form

$$\nu\lambda_0 + \sum_{j=\nu}^{n} \lambda_j = N(p) - n.$$

Since $-1 < \lambda_j < \lambda_0$, we obtain

$$\nu(\lambda_0 + 1) \leq N(p) + 1 \leq (n + 1)(\lambda_0 + 1).$$

(2.3.14) follows.

Example 2.3.15. The antipodal of the Hopf map (Example 1.4.2) is the complex Veronese map (Example 1.4.1). To show this we first note that $\mathcal{H}_3^2 = V_{Hopf} \oplus V_{Ver^{\mathbf{C}}}$ is an orthogonal direct sum. This follows by evaluating the scalar product (2.1.27) on the components of the Hopf and the complex Veronese maps (Examples 2.3.10-2.3.11).

In fact, using (2.3.11), we can choose as an orthonormal basis for $\mathcal{H}_3^2$: $1/\sqrt{3}$ times the components of the Hopf map, and $\sqrt{2/3}$ times the components of the complex Veronese map. With respect to this basis, by (2.3.12), we have

$$\langle Hopf \rangle = (2I_3) \oplus (-I_6)$$

and

$$\langle Ver^{\mathbf{C}} \rangle = (-I_3) \oplus (1/2)I_6.$$

Since $\lambda_0(\langle Hopf \rangle) = 2$, we see that (2.3.15) is satisfied as $\langle Ver^{\mathbf{C}} \rangle = -(1/2)\langle Hopf \rangle$. Thus $\langle Hopf \rangle$ and $\langle Ver^{\mathbf{C}} \rangle$ are antipodal points on $\partial\mathcal{L}_3^2$ distorted by 2. As a byproduct, evaluating traces, we also see that

$$|\langle Hopf \rangle| = 3\sqrt{2} \quad \text{and} \quad |\langle Ver^{\mathbf{C}} \rangle| = \frac{3}{\sqrt{2}}.$$

Since $1 \leq \nu(C) \leq \dim V < \dim \mathcal{H}^p$, $C \in \partial\mathcal{L}^p$, (2.3.14) immediately implies the following:

Corollary 2.3.16. *The distortion function* $\lambda_0 : \partial\mathcal{L}^p \to \mathbf{R}$ *satisfies*

$$\frac{1}{\dim \mathcal{H}^p - 1} < \lambda_0 < \dim \mathcal{H}^p - 1. \tag{2.3.17}$$

In Section 2.5 we will determine the exact dimension of $\mathcal{L}^p$. From the dimension formula for $\mathcal{L}^p$ it will follow that

$$\dim \mathcal{L}^p = \dim \mathcal{E}^p = O(\dim S^2(\mathcal{H}^p)) = O((\dim \mathcal{H}^p)^2), \text{ as } p, m \to \infty.$$

We conclude that, for $\mathcal{L}^p$, even the crude estimate (2.3.17) is significantly better than the general estimate (A.1.3) for the distortion of convex sets (Appendix 1). This indicates that the moduli $\mathcal{L}^p$ is getting significantly less distorted than a simplex as $p, m \to \infty$.

Example 2.3.17. Let $f_\xi : S^3 \to S_{W_p}$ for p odd, and $f_\xi : S^3 \to S_{R_p}$ for p even, as in Example 2.3.12. By (2.3.12), the maximal eigenvalue of $\langle f_\xi \rangle$ is $(p-1)/2$, for p odd, and p, for p even. By Theorem 2.3.14, these are also the values of the distortion of $\mathcal{L}_3^p$ at $\langle f_\xi \rangle$. In particular, we have the following distortion values

$$\lambda_0(\langle Tet \rangle) = 6 \text{ in } \mathcal{L}_3^6$$
$$\lambda_0(\langle Oct \rangle) = 8 \text{ in } \mathcal{L}_3^8$$
$$\lambda_0(\langle Ico \rangle) = 12 \text{ in } \mathcal{L}_3^{12}.$$

According to Theorem 2.3.14, the distortion function λ_0, defined on $\partial\mathcal{L}^p$, extends continuously to $\mathcal{L}^p$ as the maximal eigenvalue of the symmetric endomorphisms of $\mathcal{H}^p$ that comprise $\mathcal{L}^p$. It is well-known from linear algebra that the maximal eigenvalue satisfies

$$\lambda_0(C) = \max\{\langle C\chi, \chi\rangle \,|\, \chi \in S_{\mathcal{H}^p}\}, \quad C \in \mathcal{L}^p.$$

Since the right-hand side is convex in C, we conclude that the function $\lambda_0 : \mathcal{L}^p \to \mathbf{R}$ is *convex* (Appendix 1). In particular, the maximum $\lambda_{\max}$ of λ_0 is attained at an *extremal point* of $\mathcal{L}^p$.

Corollary 2.3.18. *The maximum distortion of the moduli $\mathcal{L}^p$ is attained at a point that corresponds to a linearly rigid p-eigenmap.*

Remark. For general convex sets, $\lambda_{\max}$ may also be attained at non-extremal points (e.g. take $\mathcal{L}$ to be a convex polygon in the plane with some parallel sides). Note also that the set of points where $\lambda_{\max}$ is attained may be disconnected (e.g. take $\mathcal{L}$ to be an isosceles triangle on the plane with interior point on the symmetry axis away from the base).

Example 2.3.15 suggests the following:

Theorem 2.3.19. *Let $f : S^m \to S_V$ be a full p-eigenmap of boundary type, $\langle f\rangle \in \partial\mathcal{L}^p$, and $f^o : S^m \to S_{V^o}$ its antipodal. Then V_f and V_{f^o} are orthogonal in the sense that*

$$V_f^{\perp} \subset V_{f^o}. \tag{2.3.18}$$

Moreover, $\dim(V_f \cap V_{f^o})$ is the number of nonmaximal and nonminimal eigenvalues of $\langle f\rangle$ in $(-1, \lambda_0(\langle f\rangle))$, or equivalently:

$$\dim(V_f \cap V_{f^o}) = \dim V - \nu(\langle f\rangle), \tag{2.3.19}$$

where $\nu(\langle f\rangle)$ is the multiplicity of the largest eigenvalue $\lambda_0(\langle f\rangle)$ of $\langle f\rangle$. Equality holds in (2.3.18) iff f has orthonormal components. In this case f^o also has orthonormal components, and $V_f \oplus V_{f^o} = \mathcal{H}^p$ is an orthogonal direct sum.

PROOF. Let $C = \langle f\rangle \in \partial\mathcal{L}^p$. By Proposition 2.3.13, $\operatorname{im}(C + I) = V_f$ and $\operatorname{im}(C^o + I) = V_{f^o}$. Since C and C^o are symmetric, $\ker(C + I) = V_f^{\perp}$ and $\ker(C^o + I) = V_{f^o}^{\perp}$. We need to show that if $\chi \in \mathcal{H}^p$ is in $\ker(C + I)$ then χ is orthogonal to all $\chi' \in \ker(C^o + I)$. We have $C\chi = -\chi$ and $C^o\chi' = -\chi'$. By (2.3.15), the latter rewrites as $C\chi' = \lambda_0\chi'$, where λ_0 is the largest eigenvalue of C. We conclude that -1 and λ_0 are distinct $(1 + \lambda_0 > 0)$ eigenvalues of C with respective eigenfunctions χ and χ'. Since C is symmetric, χ and χ' must be orthogonal. (2.3.18) follows.

The second statement follows from

$$V_f \cap V_{f^o} = \operatorname{im}(C + I) \cap \operatorname{im}\left(-\frac{1}{\lambda_0}C + I\right),$$

and the fact that the largest eigenvalue of C is λ_0 and the smallest is -1. Since f has orthonormal components iff $G(f)$ is conformal (a constant multiple of the identity), iff $\dim(V_f \cap V_{f^o}) = 0$, the last statement also follows.

Corollary 2.3.20. *Let $f : S^m \to S_V$ be a full linearly nonrigid p-eigenmap of boundary type and assume that f has orthonormal components. Then the cone $\mathcal{C}_f$ with base $\mathcal{L}_f$ and vertex $\langle f^o \rangle$ is straight and the boundary of $\mathcal{C}_f$ is entirely contained in the boundary of $\mathcal{L}^p$.*

PROOF. Since f has orthonormal components we have $V_f \oplus V_{f^o} = \mathcal{H}^p$ (Theorem 2.3.19). If $\langle f' \rangle_f \in \partial \mathcal{L}_f$ then $V_{f'} \subset V_f$ is a proper subset so that $V_{f'} + V_{f^o}$ does not give the whole of $\mathcal{H}^p$. By Theorem 2.3.5, each point on the segment connecting $\langle f' \rangle$ and $\langle f^o \rangle$ is on the boundary. To show that $\mathcal{C}_f$ is a straight cone, we need to work out the scalar product of $\iota(\langle f' \rangle_f) - \langle f \rangle$ and $\langle f^o \rangle$ in $\mathcal{E}^p$. By (2.3.4), the former has the form $A^\top \langle f' \rangle_f A$, where $f = A \circ f_p$, with $A : \mathcal{H}^p \to V$ linear and surjective. We thus have

$$\begin{aligned} \langle \iota(\langle f' \rangle_f) - \langle f \rangle, \langle f^o \rangle \rangle &= \langle A^\top \langle f' \rangle_f A, \langle f^o \rangle \rangle \\ &= \operatorname{trace}(A^\top \langle f' \rangle_f A \langle f^o \rangle). \end{aligned}$$

By Theorem 2.3.19, choosing suitable orthonormal bases, we have $f_p = (cf, c_o f^o)$ for some constants $c, c_o > 0$. Writing A in terms of these bases, we obtain that the last trace is a constant multiple of $\operatorname{trace} \langle f' \rangle_f$. This, however, is zero by Lemma 2.3.2.

Theorem 2.3.21. *With the notations of Theorem 2.3.14, we have*

$$||C|^2 - \lambda_0(C) \dim \mathcal{H}^p| \leq \frac{\dim V - \nu(C)}{2} (\lambda_0(C) + 1)^2, \tag{2.3.20}$$

where $\nu(C)$ is the multiplicity of the largest eigenvalue $\lambda_0(C)$ of $C = \langle f \rangle \in \partial \mathcal{L}^p$. In particular, if $f : S^m \to S_V$ has orthonormal components then $|C|^2 = \lambda_0(C) \dim \mathcal{H}^p$.

PROOF. Since C is traceless, we have

$$|C + I|^2 = |C|^2 + 2 \operatorname{trace} C + N(p) + 1 = |C|^2 + N(p) + 1.$$

On the other hand

$$|C + I|^2 = \sum_{j=0}^{n} (\lambda_j + 1)^2 = \nu(\lambda_0 + 1)^2 + \sum_{j=\nu}^{n} (\lambda_j + 1)^2,$$

where λ_j, $j = 0, \ldots, n$, are the eigenvalues > -1 of C in decreasing order. Combining the last two formulas, we have

$$\nu(\lambda_0 + 1)^2 \leq |C|^2 + N(p) + 1 \leq (n + 1)(\lambda_0 + 1)^2. \tag{2.3.21}$$

This is valid for any $C \in \partial \mathcal{L}^p$. We now replace C by its antipodal C^o. To do this, recall that the largest eigenvalue λ_0 of C is of multiplicity ν and the smallest eigenvalue is of multiplicity $N(p) - n$. By (2.3.15), C^o has largest

eigenvalue $1/\lambda_0$ with multiplicity $N(p)-n$ and smallest eigenvalue -1 with multiplicity ν. Thus, to replace C by C^o amounts to replace λ_0 by $1/\lambda_0$, ν by $N(p)-n$, and $n+1$ by $N(p)+1-\nu$. Using (2.3.15), we obtain

$$(N(p)-n)(\lambda_0+1)^2 \leq |C|^2+\lambda_0^2(N(p)+1) \leq (N(p)+1-\nu)(\lambda_0+1)^2, \quad (2.3.22)$$

where we multiplied through λ_0^2. Adding (2.3.21) and (2.3.22), we have

$$\begin{aligned}(N(p)-n+\nu)(\lambda_0+1)^2 &\leq 2|C|^2+(\lambda_0^2+1)(N(p)+1)\\ &\leq (N(p)+1-\nu+n+1)(\lambda_0+1)^2.\end{aligned}$$

Completing the square in λ_0^2+1, after simplification, we arrive at

$$\begin{aligned}-(n+1-\nu)(\lambda_0^2+1)^2+\lambda_0(N(p)+1) &\leq 2|C|^2\\ &\leq (n+1-\nu)(\lambda_0^2+1)^2+\lambda_0(N(p)+1).\end{aligned}$$

This is (2.3.20). The second statement is clear since f has orthonormal components iff $\nu=n+1$.

Combining the upper estimate in (2.3.14) and (2.3.20), we can obtain an upper bound for any $C \in \partial\mathcal{L}^p$. Using the Gram matrix representation (2.3.12), we can also obtain a suitable lower bound for any $C \in \mathcal{L}^p$ corresponding to a full p-eigenmap $f: S^m \to S^n$, $C=\langle f\rangle$.

Theorem 2.3.22. *Let $f: S^m \to S_V = S^n$, $V=\mathbf{R}^{n+1}$, be a full p-eigenmap, $C=\langle f\rangle \in \mathcal{L}^p$. Let*

$$c_f = \min_{0\leq j<j'\leq n} \cos^2 \alpha_{jj'},$$

where $\alpha_{jj'}$ is the angle between f^j and $f^{j'}$ in $\mathcal{H}^p$. We have

$$\dim \mathcal{H}^p\left(\dim \mathcal{H}^p \frac{(\dim V-1)c_f+1}{\dim V}-1\right) \leq |C|^2. \quad (2.3.23)$$

If, in addition, f is of boundary type, $C=\partial\mathcal{L}^p$, then we have

$$|C|^2 \leq \dim \mathcal{H}^p\left(\dim \mathcal{H}^p \frac{\dim V+\nu(C)}{2\nu(C)^2}-1\right). \quad (2.3.24)$$

If $f: S^m \to S_V$ has orthonormal components then equalities hold in (2.3.23)-(2.3.24).

Proof. As noted above, the upper estimate (2.3.24) follows from Theorems 2.3.14 and 2.3.21. To derive the lower estimate (2.3.23), we will make use of the equation

$$\sum_{j=0}^{n} |f^j|^2 = N(p)+1, \quad (2.3.25)$$

a consequence of the fact that $\sum_{j=0}^n (f^j)^2 = 1$ on S^m. We compute

$$|C|^2+N(p)+1 = |C+I|^2 = |G(f)|^2 = \sum_{j,j'=0}^{n} \langle f^j, f^{j'}\rangle^2$$

$$
\begin{aligned}
&= \sum_{j=0}^{n} |f^j|^4 + 2 \sum_{0\le j<j'\le n} |f^j|^2 |f^{j'}|^2 \cos^2 \alpha_{jj'} \\
&\ge \sum_{j=0}^{n} |f^j|^4 + 2c_f \sum_{0\le j<j'\le n} |f^j|^2 |f^{j'}|^2 \\
&= \sum_{j=0}^{n} |f^j|^4 + c_f \sum_{j=0}^{n} |f^j|^2 (N(p) + 1 - |f^j|^2) \\
&= (1 - c_f) \sum_{j=0}^{n} |f^j|^4 + (N(p)+1)^2 c_f.
\end{aligned}
$$

This attains its minimum in the variables $|f^j|^2$, $j = 0, \ldots, n$, subject to (2.3.25) when $|f^0|^2 = \ldots = |f^n|^2 = (N(p)+1)/(n+1)$. The lower estimate (2.3.23) follows.

Given a compact convex set $\mathcal{L}$ in a Euclidean vector space $\mathcal{E}$, assume that the origin is in the interior of $\mathcal{L}$. We denote by $r(\mathcal{L})$ and $R(\mathcal{L})$ the *inradius* and the *circumradius* of $\mathcal{L}$:

$$
r(\mathcal{L}) = \min \{|C| \,|\, C \in \partial \mathcal{L}\} \quad \text{and} \quad R(\mathcal{L}) = \max \{|C| \,|\, C \in \mathcal{L}\}.
$$

Corollary 2.3.23. *We have*

$$
(1 <) \sqrt{\frac{\dim \mathcal{H}^p}{\dim \mathcal{H}^p - 1}} \le r(\mathcal{L}^p) \le R(\mathcal{L}^p) < \sqrt{\dim \mathcal{H}^p (\dim \mathcal{H}^p - 1)}. \quad (2.3.26)
$$

PROOF. The lower estimate for $r(\mathcal{L}^p)$ follows from (2.3.23) by setting $\dim V = \dim \mathcal{H}^p - 1$. To prove the upper estimate, we will use the notations in the proof of Theorem 2.3.22. We compute

$$
\begin{aligned}
|C|^2 + N(p) + 1 = |C + I|^2 = |G(f)|^2 &= \sum_{j,j'=0}^{n} \langle f^j, f^{j'} \rangle^2 \\
&= \sum_{j=0}^{n} |f^j|^4 + 2 \sum_{0\le j<j'\le n} |f^j|^2 |f^{j'}|^2 \cos^2 \alpha_{jj'} \\
&\le \sum_{j=0}^{n} |f^j|^4 + 2 \sum_{0\le j<j'\le n} |f^j|^2 |f^{j'}|^2 \\
&= \left(\sum_{j=0}^{n} |f^j|^2 \right)^2 = (N(p)+1)^2.
\end{aligned}
$$

The corollary follows.

The last result in this section follows directly from the lower estimate (2.3.23). It implies that, up to scaling, $\sqrt{c_f}$ is a lower bound for $|C|$, when C is in the interior of the moduli $\mathcal{L}^p$.

Corollary 2.3.24. *For C in the interior of $\mathcal{L}^p$, we have*

$$|C| \geq \sqrt{c_f}\sqrt{\dim \mathcal{H}^p(\dim \mathcal{H}^p - 1)}. \tag{2.3.27}$$

2.4 Raising and Lowering the Degree

Let $f : S^m \to S_V$ be a p-eigenmap. We define the harmonic $(p \pm 1)$-homogeneous polynomial maps

$$f^{\pm} : \mathbf{R}^{m+1} \to V \otimes \mathcal{H}^1$$

by

$$f^+(a) = \sqrt{\frac{\lambda_{2p}}{2\lambda_p}}\delta_a f \quad \text{and} \quad f^-(a) = \sqrt{\frac{2}{\lambda_{2p}}}\partial_a f, \ a \in \mathbf{R}^{m+1}.$$

Here, as usual, $\mathcal{H}^1 = (\mathbf{R}^{m+1})^*$, and $f : \mathbf{R}^{m+1} \to V$ is considered as a harmonic p-homogeneous polynomial map, and, in $\delta_a f = H(a^* f)$ and $\partial_a f$ (Section 2.1), the harmonic projection operator H and ∂_a act on f componentwise. With respect to the standard basis $\{e_i\}_{i=0}^m \subset \mathbf{R}^{m+1}$, we have

$$f^+ = \sqrt{\frac{\lambda_{2p}}{2\lambda_p}}\sum_{i=0}^m \delta_i f \otimes y_i \quad \text{and} \quad f^- = \sqrt{\frac{2}{\lambda_{2p}}}\sum_{i=0}^m \partial_i f \otimes y_i, \tag{2.4.1}$$

where, $e_i^* = y_i$, $i = 0, \ldots, m$, are the elements of the dual basis in $\mathcal{H}^1$. (We use here the variable $y \in \mathbf{R}^{m+1}$ to distinguish it from the natural variable x of f.) Note that $f^{\pm}$ may not be full even if f is (Problem 2.17).

Proposition 2.4.1. *$f^{\pm}$ are spherical so that the restrictions $f^{\pm} : S^m \to S_{V \otimes \mathcal{H}^1}$ are $(p \pm 1)$-eigenmaps.*

Before the proof we derive some useful formulas.

Let $f_1, f_2 : \mathbf{R}^{m+1} \to V$ be harmonic p-homogeneous polynomial maps. We have the following identities

$$\sum_{i=0}^m \langle \partial_i f_1, \partial_i f_2 \rangle = \frac{1}{2}\triangle\langle f_1, f_2 \rangle, \tag{2.4.2}$$

and

$$\sum_{i=0}^m \langle \delta_i f_1, \delta_i f_2 \rangle = (m-1)\frac{2p}{\lambda_{2p}}\langle f_1, f_2 \rangle \cdot \rho^2 + \frac{2p^2}{\lambda_{2p}^2}\triangle\langle f_1, f_2 \rangle \cdot \rho^4. \tag{2.4.3}$$

(2.4.2) follows immediately from harmonicity of f_1 and f_2. To show (2.4.3), we use (2.1.8), and compute

$$\begin{aligned}\sum_{i=0}^{m}\langle\delta_i f_1,\delta_i f_2\rangle &= \sum_{i=0}^{m}\left\langle x_i f_1-\frac{2p}{\lambda_{2p}}\partial_i f_1\rho^2, x_i f_2-\frac{2p}{\lambda_{2p}}\partial_i f_2\rho^2\right\rangle\\ &= \langle f_1,f_2\rangle\rho^2-\frac{2p}{\lambda_{2p}}\langle\partial_x f_1,f_2\rangle\rho^2-\frac{2p}{\lambda_{2p}}\langle f_1,\partial_x f_2\rangle\rho^2\\ &+\frac{4p^2}{\lambda_{2p}^2}\sum_{i=0}^{m}\langle\partial_i f_1,\partial_i f_2\rangle\rho^4,\end{aligned}$$

where $\partial_x=\sum_{i=0}^{m}x_i\partial_i$ is the radial derivative. By homogeneity, $\partial_x f_1=pf_1$ and $\partial_x f_2=pf_2$. (2.4.2) now gives (2.4.3).

PROOF OF PROPOSITION 2.4.1. Since f is spherical, we have $|f|^2=\rho^{2p}$ as polynomials. Taking the Laplacian of both sides of this equation and using (2.4.2), we obtain

$$\sum_{i=0}^{m}|\partial_i f|^2=\frac{1}{2}\triangle\rho^{2p}=\frac{\lambda_{2p}}{2}\rho^{2(p-1)}.$$

The last equality is because $\triangle\rho^{2p}=\lambda_{2p}\rho^{2(p-1)}$ (as $\rho^{2p}=1$ on S^m; see also (2.1.2) with $k=p$ and $\xi=1$). Restricting to S^m, we obtain that f^- is spherical. Using this and (2.4.3) we now have

$$\begin{aligned}\sum_{i=0}^{m}|\delta_i f|^2 &= (m-1)\frac{2p}{\lambda_{2p}}|f|^2\rho^2+\frac{2p^2}{\lambda_{2p}^2}\triangle|f|^2\cdot\rho^4\\ &= \frac{p+m-1}{2p+m-1}\rho^{2(p+1)}=\frac{2\lambda_p}{\lambda_{2p}}\rho^{2(p+1)}.\end{aligned}$$

Restricting to S^m again, sphericality of f^+ follows.

Remark. Raising and lowering the degree was used by Toth [4] and later by Gauchman-Toth [1] as a technical tool to solve the DoCarmo-Wallach problem. This will be discussed in Chapter 3. Degree-lowering for (locally defined) spherical minimal immersions also appears implicitly in Wallach [1] (Lemma 6.3).

For the standard minimal immersion $f_p: S^m\to S_{\mathcal{H}^p}$, we have $f_p^{\pm}: S^m\to S_{\mathcal{H}^p\otimes\mathcal{H}^1}$. Since the Laplacian commutes with the isometries on S^m, it also commutes with the harmonic projection operator H. It follows that $f_p^{\pm}$ are $SO(m+1)$-equivariant with respect to the $SO(m+1)$-module structure of the tensor product $\mathcal{H}^p\otimes\mathcal{H}^1$. The images of $f_p^{\pm}$ actually span the $SO(m+1)$-submodules $\mathcal{H}^{p\pm1}$ in the tensor product $\mathcal{H}^p\otimes\mathcal{H}^1$. For the next lemma, recall from Section 2.1 that $\iota_{\pm}:\mathcal{H}^{p\pm1}\to\mathcal{H}^p\otimes\mathcal{H}^1$ are $SO(m+1)$-equivariant isometric linear imbeddings of $\mathcal{H}^{p\pm1}$ into $\mathcal{H}^p\otimes\mathcal{H}^1$.

Lemma 2.4.2. *We have*

$$f_p^{\pm}(x) = \iota_{\pm}(f_{p\pm1}(x)),\ x \in S^m, \tag{2.4.4}$$

in particular, $f_p^{\pm}$ (made full) is standard.

PROOF. We fix orthonormal bases $\{f_p^j\}_{j=0}^{N(p)} \subset \mathcal{H}^p$ and $\{f_{p-1}^l\}_{l=0}^{N(p-1)} \subset \mathcal{H}^{p-1}$. Using Lemma 2.1.4, we compute

$$\begin{aligned}
\iota_-(f_{p-1}(x)) &= \sum_{l=0}^{N(p-1)} f_{p-1}^l(x)\iota_-(f_{p-1}^l) \\
&= c_p^- \sum_{i=0}^{m}\sum_{l=0}^{N(p-1)} f_{p-1}^l(x)\ \delta_i f_{p-1}^l \otimes y_i \\
&= c_p^- \sum_{i=0}^{m}\sum_{l=0}^{N(p-1)} f_{p-1}^l(x) \sum_{j=0}^{N(p)} \langle \delta_i f_{p-1}^l, f_p^j\rangle f_p^j \otimes y_i \\
&= \frac{c_p^-}{\mu_{p-1}} \sum_{i=0}^{m}\sum_{l=0}^{N(p-1)} f_{p-1}^l(x) \sum_{j=0}^{N(p)} \langle f_{p-1}^l, \partial_i f_p^j\rangle f_p^j \otimes y_i \\
&= \frac{c_p^-}{\mu_{p-1}} \sum_{i=0}^{m}\sum_{j=0}^{N(p)} (\partial_i f_p^j)(x) f_p^j \otimes y_i \\
&= f_p^-(x),
\end{aligned}$$

where we used (2.1.21) and (2.1.24). Our formula (2.4.4) follows for f^-. The computation for f^+ are analogous.

Remark. Using Corollary 2.3.4, the congruences $f_p^{\pm} \cong f_{p\pm1}$ follow from the $SO(m+1)$-equivariance of $f_p^{\pm}$. Indeed, $SO(m+1)$-equivariance translates into $\langle f_p^{\pm}\rangle \in \mathcal{L}^{p\pm1}$ being left fixed by $SO(m+1)$. Thus $\langle f_p^{\pm}\rangle$ correspond to the origin.

Let $f : S^m \to S_V$ be a full p-eigenmap. Setting $f = A \circ f_p$, by definition, we have $f^{\pm} = (A \otimes I) f_p^{\pm}$. Using (2.4.4), we have

$$\begin{aligned}
\langle f^{\pm}\rangle &= \iota_{\pm}^{\top}(A^{\top} \otimes I)(A \otimes I)\iota_{\pm} - I \\
&= \iota_{\pm}^{\top}((A^{\top}A - I) \otimes I)\iota_{\pm} \\
&= \iota_{\pm}^{\top}(\langle f\rangle \otimes I)\iota_{\pm}.
\end{aligned}$$

In view of this, we define the *degree-raising and -lowering operators*

$$\Phi^{\pm} = \Phi_p^{\pm} : S^2(\mathcal{H}^p) \to S^2(\mathcal{H}^{p\pm1})$$

by

$$\Phi^{\pm}(C) = \iota_{\pm}^{\top} \circ (C \otimes I) \circ \iota_{\pm}.$$

Clearly, $\Phi_p^{\pm}$ are homomorphisms of $SO(m+1)$-modules. The previous computation amounts to

$$\Phi^{\pm}(\langle f \rangle) = \langle f^{\pm} \rangle. \tag{2.4.5}$$

In particular, we have

$$\Phi^{\pm}(\mathcal{L}^p) \subset \mathcal{L}^{p\pm 1}.$$

2.5 Exact Dimension of the Moduli $\mathcal{L}^p$

We are now ready to describe the $SO(m+1)$-module structure of $\mathcal{E}^p$.

Theorem 2.5.1. *The symmetric square $S^2(\mathcal{H}^p)$ contains $\mathcal{P}^{2p}$ as an $SO(m+1)$-submodule and*

$$\mathcal{E}^p \cong S^2(\mathcal{H}^p)/\mathcal{P}^{2p}. \tag{2.5.1}$$

In particular, by (2.1.5), we have

$$\begin{aligned} \dim \mathcal{L}^p &= \dim \mathcal{E}^p \\ &= \binom{\binom{p+m}{m} - \binom{p+m-2}{m} + 1}{2} - \binom{2p+m}{m}. \end{aligned} \tag{2.5.2}$$

Corollary 2.5.2. *If $f : S^2 \to S_V$ is a full p-eigenmap then $\dim V = 2p$ and f is congruent to the standard minimal immersion.*

Remark. Theorem 2.5.1 can be found implicitly in DoCarmo-Wallach [1] (4.4. Lemma, p.51) who used higher fundamental forms for their proof. Our elementary exposition below is new. Corollary 2.5.2 is due to Calabi [1] and DoCarmo-Wallach [2]. (For more details of this approach, see Wallach [1], p.33.)

Proof of Corollary 2.5.2. For $m = 2$, we have

$$\dim \mathcal{P}_3^{2p} = \binom{2p+2}{2} = S^2(\mathcal{H}_2^p)$$

since $\dim \mathcal{H}_2^p = 2p+1$. By (2.5.1), $\mathcal{E}_2^p$ reduces to a point.

By Theorem 2.5.1, $\mathcal{L}_m^p$ is nontrivial iff $m \geq 3$ and $p \geq 2$. The first nonrigid range corresponds to $m = 3$ and $p = 2$, and, again by Theorem 2.5.1, the moduli space $\mathcal{L}_3^2$ is 10-dimensional.

A key step in the proof of Theorem 2.5.1 is to encode the information as to how far a full harmonic p-homogeneous polynomial map $f : \mathbf{R}^{m+1} \to V$ is from being spherical. We do this by associating to f the polynomial

$$\Psi^0(f) = |f|^2 - |f_p|^2 = |f|^2 - \rho^{2p} \in \mathcal{P}^{2p}.$$

Setting, as usual, $f = A \circ f_p$ with $A : \mathcal{H}^p \to V$ linear and surjective, we have

$$\begin{aligned}\Psi^0(f) &= |A \circ f_p|^2 - |f_p|^2 \\ &= \langle A^\top A f_p, f_p\rangle - \langle f_p, f_p\rangle \\ &= \langle C f_p, f_p\rangle,\end{aligned}$$

where $C = \langle f\rangle = A^\top A - I \in S^2(\mathcal{H}^p)$ is the point corresponding to f (see the discussion after Corollary 2.3.4). This suggests that we consider the linear extension

$$\Psi^0 : S^2(\mathcal{H}^p) \to \mathcal{P}^{2p}$$

defined by

$$\Psi^0(C) = \langle C f_p, f_p\rangle, \quad C \in S^2(\mathcal{H}^p).$$

Clearly, Ψ^0 is a homomorphism of $SO(m+1)$-modules. By definition, $\Psi^0(f) = \Psi^0(\langle f\rangle)$, and we have

$$\ker \Psi^0 = \mathcal{E}^p.$$

This reduces Theorem 2.5.1 to the following:

Theorem 2.5.3. *The homomorphism*

$$\Psi^0 : S^2(\mathcal{H}^p) \to \mathcal{P}^{2p}$$

is onto.

Example 2.5.4. For $p = 1$ the standard minimal immersion $f_1 : S^m \to S^m$ is the identity. Hence $\Psi^0 : S^2(\mathcal{H}^1) \to \mathcal{P}^2$ associates to $C \in S^2(\mathcal{H}^1)$ the quadratic form $\Psi^0(C) = \langle Cx, x\rangle$. Thus, in this case, Ψ^0 is actually an isomorphism. This fact, combined with (2.1.14), gives the isomorphism

$$S^2(\mathcal{H}^1) \cong \mathcal{P}^2 \cong \mathcal{H}^0 \oplus \mathcal{H}^2.$$

Under this isomorphism the projection $S^2(\mathcal{H}^1) \to \mathcal{H}^0$ is defined by the trace.

Remark. There is a very simple algebraic interpretation of Ψ^0 as follows. Let $\mathcal{P} = \mathcal{P}_{m+1}$ denote the space of all homogeneous polynomials in $m+1$ variables. $\mathcal{P}$ is a graded algebra, where the grading is given by the degree. As usual, $\mathcal{P}^p \subset \mathcal{P}$ denotes the linear subspace of homogeneous polynomials of degree p. $\mathcal{P}$ is an $SO(m+1)$-module with the module structure that preserves the grading. (The $SO(m+1)$-module structure on $\mathcal{P}^p$ was discussed in Section 2.1.)

We let $\Pi^0 : \mathcal{P} \otimes \mathcal{P} \to \mathcal{P}$ denote the homomorphism given by *multiplication*:

$$\Pi^0(\xi \otimes \eta) = \xi \cdot \eta, \quad \xi, \eta \in \mathcal{P}.$$

We have $\Pi^0(\mathcal{P}^p \otimes \mathcal{P}^q) \subset \mathcal{P}^{p+q}$. We denote the restriction $\Pi^0|_{\mathcal{P}^p \otimes \mathcal{P}^q}$ by $\Pi^0_{p,q}$. With this, we have

$$\Pi^0_{p,p}|_{S^2(\mathcal{H}^p)} = \Psi^0,$$

where $S^2(\mathcal{H}^p)$ is considered as a linear subspace in $\mathcal{H}^p \otimes \mathcal{H}^p \subset \mathcal{P}^p \otimes \mathcal{P}^p$. The simplest way to see this is by writing everything in coordinates. Let $\{f_p^j\}_{j=0}^{N(p)} \subset \mathcal{H}^p$ be an orthonormal basis, and $\sum_{j=0}^{N(p)} c_{jl} f_p^j \otimes f_p^l$ a typical element of $S^2(\mathcal{H}^p)$ with matrix $C = (c_{jl})_{j,l=0}^{N(p)}$, $c_{jl} = c_{lj}$. We have

$$\begin{aligned}
\Psi^0(C)(x) &= \langle C f_p(x), f_p(x) \rangle \\
&= \left\langle C \sum_{j=0}^{N(p)} f_p^j(x) f_p^j, \sum_{l=0}^{N(p)} f_p^l(x) f_p^l \right\rangle \\
&= \sum_{j,l=0}^{N(p)} \langle C f_p^j, f_p^l \rangle f_p^j(x) f_p^l(x) \\
&= \sum_{j,l=0}^{N(p)} c_{jl} f_p^j(x) f_p^l(x) \\
&= \Pi^0_{p,p} \left(\sum_{j,l=0}^{N(p)} c_{jl} f_p^j \otimes f_p^l \right)(x).
\end{aligned}$$

The claim follows.

Returning to the proof of Theorem 2.5.3, we first show the following:

Lemma 2.5.5. *$\mathcal{H}^{2p} \subset \mathcal{P}^{2p}$ is in the image of Ψ^0.*

PROOF. Let $f : \mathbf{R}^{m+1} \to \mathbf{R}$ be given by $f(x) = H(x_m^p)$, $x = (x_0, \ldots, x_m) \in \mathbf{R}^{m+1}$. Since $\mathcal{H}^{2p}$ is irreducible, it is enough to prove that the harmonic part of $\Psi^0(f)$ is nonzero. Using the definition of Ψ^0, we have

$$\begin{aligned}
H(\Psi^0(f)) &= H(H(x_m^p)^2 - \rho^{2p}) \\
&= H(H(x_m^p)^2) \\
&= H(x_m^{2p})
\end{aligned}$$

since, by (2.1.9), $H(x_m^p) \equiv x_m^p (\bmod \rho^2)$ and $H(\rho^2 \cdot) = 0$, by definition. By (2.1.9) again, $H(x_m^{2p})$ is nonzero. The lemma follows.

We prove Theorem 2.5.3 by induction with respect to p. We accomplished the first step ($p = 1$) in Example 2.5.4. To perform the general induction step, we will bring in the degree-raising and -lowering operators $\Phi_p^{\pm}$ introduced at the end of the previous section. We write Ψ_p^0 to indicate the dependence of Ψ^0 on p.

Theorem 2.5.6. *For $C \in S^2(\mathcal{H}^p)$, we have*

$$\Psi^0_{p+1}(\Phi^+_p(C)) = (m-1)\frac{p}{\lambda_p}\Psi^0_p(C)\rho^2 + \frac{p^2}{\lambda_p\lambda_{2p}}\triangle(\Psi^0_p(C))\rho^4. \tag{2.5.3}$$

PROOF. First of all we rewrite the left-hand side of (2.5.3) as

$$\begin{aligned}\Psi^0_{p+1}(\Phi^+_p(C)) &= \langle \Phi^+_p(C)f_p, f_p\rangle \\ &= \langle (C\otimes I)f^+_p, f^+_p\rangle \\ &= \frac{\lambda_{2p}}{2\lambda_p}\sum_{i=0}^{m}\langle \delta_i C f_p, \delta_i f_p\rangle,\end{aligned}$$

where we used (2.4.4). We now use (2.4.3) with $f_1 = Cf_p$ and $f_2 = f_p$, and obtain

$$\sum_{i=0}^{m}\langle \delta_i(Cf_p), \delta_i f_p\rangle = (m-1)\frac{2p}{\lambda_{2p}}\langle Cf_p, f_p\rangle\rho^2 + \frac{2p^2}{\lambda^2_{2p}}\triangle\langle Cf_p, f_p\rangle\rho^4.$$

(2.5.3) follows since $\Psi^0(C) = \langle Cf_p, f_p\rangle$.

PROOF OF THEOREM 2.5.3. We proceed by induction with respect to p. For the general induction step, we assume that $\Psi^0_p : S^2(\mathcal{H}^p) \to \mathcal{P}^{2p}$ is onto. By Lemma 2.5.5, $\mathcal{H}^{2(p+1)}$ is in the image of Ψ^0_{p+1}. In view of (2.1.1), it is enough to show that $\mathcal{P}^{2p}\rho^2$ is in the image of Ψ^0_{p+1}. Let $\eta \in \mathcal{P}^{2p}$. We need to show that $\eta\rho^2$ is in the image of Ψ^0_{p+1}. To do this, we apply Lemma 2.1.2 with

$$c_0 = (m-1)\frac{p}{\lambda_p} \quad \text{and} \quad c_1 = \frac{p^2}{\lambda_p\lambda_{2p}}$$

to obtain $\xi \in \mathcal{P}^{2p}$ satisfying

$$c_0\xi + c_1\triangle\xi\cdot\rho^2 = \eta.$$

By the induction hypothesis, there exists $C \in S^2(\mathcal{H}^p)$ such that $\Psi^0_p(C) = \xi$. Retracing our steps and using (2.5.3), we obtain

$$\Psi^0_{p+1}(\Phi^+_p(C)) = \eta\rho^2.$$

Theorem 2.5.3 and, hence, Theorem 2.5.1 follow.

Remark 1. For $C \in S^2(\mathcal{H}^p)$, we have

$$\triangle(\Psi^0_p(C)) = \lambda_{2p}\Psi^0_{p-1}(\Phi^-_p(C)). \tag{2.5.4}$$

This can be deduced from (2.4.2) along the same lines as (2.5.3).

Using the comparison formula for the Euclidean and spherical Laplacians, for $\Psi_p^0(C) \in \mathcal{P}^{2p}$, $C \in S^2(\mathcal{H}^p)$, we have

$$\triangle(\Psi_p^0(C)) = \left(\lambda_{2p} I - \triangle^{S^m}\right) \Psi_p^0(C).$$

Restricting (2.5.3) and (2.5.4) to S^m, we obtain

$$\Psi_{p+1}^0(\Phi_p^+(C)) = \left(I - \frac{p^2}{\lambda_p \lambda_{2p}} \triangle^{S^m}\right) \Psi_p^0(C),$$

and

$$\Psi_{p-1}^0(\Phi_p^-(C)) = \left(I - \frac{1}{\lambda_{2p}} \triangle^{S^m}\right) \Psi_p^0(C).$$

These formulas express the commutativity of the diagrams

$$\begin{array}{ccc} S^2(\mathcal{H}^p) & \xrightarrow{\Phi_p^{\pm}} & S^2(\mathcal{H}^{p\pm 1}) \\ \Psi_p^0 \downarrow & & \downarrow \Psi_{p\pm 1}^0 \\ \mathcal{P}^{2p} & \xrightarrow{\bar{\Phi}_p^{\pm}} & \mathcal{P}^{2(p\pm 1)} \end{array}$$

where the invariant differential operators $\bar{\Phi}_p^{\pm}$ are given by

$$\bar{\Phi}_p^+ = I - \frac{p^2}{\lambda_p \lambda_{2p}} \triangle^{S^m},$$

and

$$\bar{\Phi}_p^- = I - \frac{1}{\lambda_{2p}} \triangle^{S^m}.$$

The eigenvalues of $\triangle^{S^m}$ on $\mathcal{P}^{2p} = \sum_{j=0}^p \mathcal{H}^{2j}$ are λ_{2j}, $j = 0, \ldots, p$. Since $p^2/\lambda_p < 1$, we see that $\bar{\Phi}_p^+$ is injective. It is also clear that $\bar{\Phi}_p^-$ is surjective with kernel $\mathcal{H}^{2p} \subset \mathcal{P}^{2p}$.

Remark 2. In Section 2.1 we saw that the zonal spherical harmonic $H(x_m^p)$ can be expressed in terms of the ultraspherical polynomial $C_p^{(m-1)/2}$. Using deeper properties of these polynomials, a simple proof of Theorem 2.5.3 can be given as follows: In the remark after Example 2.5.4 we showed that $\Psi_p^0 : S^2(\mathcal{H}^p) \to \mathcal{P}^{2p}$ is given by the multiplication operator $\Pi_{p,p}^0$. We let $\mathcal{H}^p \cdot \mathcal{H}^p \subset \mathcal{P}^{2p}$ denote the image of $\Psi^0 = \Pi_{p,p}^0|_{S^2(\mathcal{H}^p)}$. By definition, $\mathcal{H}^p \cdot \mathcal{H}^p$ is the $SO(m+1)$-submodule of $\mathcal{P}^{2p}$ consisting of sums of products of spherical harmonics of order p. Theorem 2.5.3 asserts that $\mathcal{H}^p \cdot \mathcal{H}^p = \mathcal{P}^{2p}$. To give another proof of this, we first observe that $H(x_m^p)^2$ certainly belongs to $\mathcal{H}^p \cdot \mathcal{H}^p$. By (2.1.11), up to a constant multiple, $H(x_m^p)^2$ (restricted to S^m) is $C_p^{(m-1)/2}(x_m)$. On the other hand, expanding with respect to the ultraspherical polynomials, we have the decomposition

$$C_p^{(m-1)/2}(t)^2 = \sum_{j=0}^p c_j C_{2j}^{(m-1)/2}(t).$$

(Notice that odd-degree ultraspherical polynomials cannot occur.) If the coefficient c_j is nonzero then the orthogonal projection $\mathcal{P}^{2p} \to \mathcal{H}^{2j}$ restricted to $\mathcal{H}^p \cdot \mathcal{H}^p$ is onto $\mathcal{H}^{2j}$ since it contains $C_{2j}^{(m-1)/2}$ and $\mathcal{H}^{2j}$ is irreducible. Hence, $\mathcal{H}^{2j}$ is a component of $\mathcal{H}^p \cdot \mathcal{H}^p$. Thus, to complete the proof we need to show that all coefficients c_j, $j = 0, \ldots, p$, are nonzero. This however is a consequence of the formula

$$\begin{aligned} C_d^a(z)C_{d'}^a(z) = &\sum_{k=0}^{\min(d,d')} \frac{(d+d'+a-2k)}{(d+d'+a-k)} \\ &\times \frac{(a)_k(a)_{d-k}(a)_{d'-k}(2a)_{d+d'-k}}{k!(d-k)!(d'-k)!(a)_{d+d'-k}} \\ &\times \frac{(d+d'-2k)!}{(2a)_{d+d'-2k}} C_{d+d'-2k}^a(z), \end{aligned}$$

where $(a)_k = \Gamma(a+k)/\Gamma(a)$. (This so-called Dougall's formula is a special case of "linearization of products" for Jacobi polynomials. This problem has a long and interesting history; for details, see Askey [1], pp. 39-40.)

Let $f : S^m \to S_V$ be a full p-eigenmap. Recall that f is linearly rigid iff $\mathcal{L}_f$ is trivial. By (2.3.7), f is linearly rigid iff $S^2(V_f)$ and $\mathcal{E}^p$ have trivial intersection in $S^2(\mathcal{H}^p)$. By Theorem 2.5.1, we have

$$\dim S^2(\mathcal{H}^p)/\mathcal{E}^p = \dim \mathcal{P}^{2p} = \binom{m+2p}{m}.$$

Thus, if $f : S^m \to S_V$ is linearly rigid, then

$$\frac{\dim V(\dim V+1)}{2} = \dim S^2 V = \dim S^2(V_f) \leq \dim \mathcal{P}^{2p} = \binom{m+2p}{m}.$$

Let m be fixed. For $p \to \infty$, we have

$$\binom{m+2p}{m} = \frac{(2p+m)(2p+m-1)\ldots(2p+1)}{m!} = O(p^m).$$

Thus, if f is linearly rigid then $\dim V \leq O(p^{m/2})$ as $p \to \infty$.

In contrast, for any full p-eigenmap $f : S^m \to S_V$, we have $V_f \subset \mathcal{H}^p$, so that $\dim V = \dim V_f \leq \dim \mathcal{H}^p = O(p^{m-1})$ as $p \to \infty$.

2.6 Equivariant Imbedding of Moduli

The main result of this section is the following:

Theorem 2.6.1. *The degree-raising operator $\Phi_p^+ : \mathcal{E}^p \to \mathcal{E}^{p+1}$ is injective, and its restriction defines an $SO(m+1)$-equivariant imbedding of the standard moduli space $\mathcal{L}^p$ into $\mathcal{L}^{p+1}$. Up to a constant multiple, the degree-lowering operator $\Phi_p^- : \mathcal{E}^p \to \mathcal{E}^{p-1}$ is the transpose of Φ_{p-1}^+, and therefore it is surjective.*

Remark. Under the equivariant imbedding, the boundary points of $\mathcal{L}^p$ may not map to boundary points of $\mathcal{L}^{p+1}$. This is certainly the case for boundary points corresponding to full eigenmaps with high (but not maximal) range dimension (Problem 2.18).

For the same reason, the restriction $\Phi_p^- : \mathcal{L}^p \to \mathcal{L}^{p-1}$ is, in general, *not* surjective.

Here, we introduce the differential operator

$$D : \mathcal{H}^p \otimes \mathcal{H}^q \to \mathcal{H}^{p-1} \otimes \mathcal{H}^{q-1}$$

defined by

$$D = \sum_{i=0}^{m} \partial_i \otimes \partial_i.$$

For $p = q$, $S^2(\mathcal{H}^p) \subset \mathcal{H}^p \otimes \mathcal{H}^p$, so that D can be restricted to $\mathcal{L}^p \subset \mathcal{E}^p \subset S^2(\mathcal{H}^p)$. We claim that the restriction $D|_{\mathcal{L}^p}$ is essentially given by the degree-lowering operator.

Remark. The differential operator D was introduced in DoCarmo-Wallach [1] (Appendix, p. 57) for the purpose of deriving a recurrence formula for the irreducible decomposition of the tensor product $\mathcal{H}^p \otimes \mathcal{H}^q$. The geometric content of D acting on eigenmaps has been recognized in Toth [4].

Theorem 2.6.2. *Let $f : S^m \to S_V$ be a full p-eigenmap. We have*

$$D(\langle f \rangle) = \frac{\lambda_{2p}}{2} \langle f^- \rangle$$

and

$$D^\top(\langle f \rangle) = (p+1)^2 \frac{\lambda_{2p}}{2\lambda_p} \langle f^+ \rangle.$$

PROOF. By (2.4.5), it is enough to show that, on $S^2(\mathcal{H}^p)$,

$$D = \frac{\lambda_{2p}}{2} \Phi_p^- \tag{2.6.1}$$

and

$$D^\top = (p+1)^2 \frac{\lambda_{2p}}{2\lambda_p} \Phi_p^+. \tag{2.6.2}$$

Fix orthonormal bases $\{f_p^j\}_{j=0}^{N(p)} \subset \mathcal{H}^p$ and $\{f_{p-1}^l\}_{l=0}^{N(p-1)} \subset \mathcal{H}^{p-1}$, and let $C = (c_{jj'})_{j,j'=0}^{N(p)} \in S^2(\mathcal{H}^p)$. We compute

$$D(C) = D\left(\sum_{j,j'=0}^{N(p)} c_{jj'} f_p^j \otimes f_p^{j'} \right)$$

$$
\begin{aligned}
&= \sum_{i=0}^{m} \sum_{j,j'=0}^{N(p)} c_{jj'} \partial_i f_p^j \otimes \partial_i f_p^{j'} \\
&= \sum_{i=0}^{m} \sum_{j,j'=0}^{N(p)} \sum_{l,l'=0}^{N(p-1)} c_{jj'} \langle \partial_i f_p^j, f_{p-1}^l \rangle \langle \partial_i f_p^{j'}, f_{p-1}^{l'} \rangle f_{p-1}^l \otimes f_{p-1}^{l'} \\
&= \mu_{p-1}^2 \sum_{i=0}^{m} \sum_{l,l'=0}^{N(p-1)} \langle C\delta_i f_{p-1}^l, \delta_i f_{p-1}^{l'} \rangle f_{p-1}^l \otimes f_{p-1}^{l'} \\
&= \mu_{p-1} \sum_{i=0}^{m} \sum_{l,l'=0}^{N(p-1)} \langle \partial_i C \delta_i f_{p-1}^l, f_{p-1}^{l'} \rangle f_{p-1}^l \otimes f_{p-1}^{l'},
\end{aligned}
$$

where we used Lemma 2.1.4 several times. Thus, on $\mathcal{H}^{p-1}$:

$$
D(C) = \mu_{p-1} \sum_{i=0}^{m} \partial_i C \delta_i. \tag{2.6.3}
$$

On the other hand, by Proposition 2.1.3, we have on $\mathcal{H}^{p-1}$:

$$
\Phi_p^-(C) = \iota_-^\top (C \otimes I) \iota_- = \frac{p\lambda_{2(p-1)}}{\lambda_{2p}\lambda_{p-1}} \sum_{i=0}^{m} \partial_i C \delta_i.
$$

Comparing this with (2.6.3), (2.6.1) follows.

Finally, we prove (2.6.2) by showing that, up to a constant multiple, Φ_p^- is the transpose of Φ_{p-1}^+. We compute, for $C \in S^2(\mathcal{H}^p)$ and $C' \in S^2(\mathcal{H}^{p-1})$:

$$
\begin{aligned}
\langle \Phi_p^-(C), C' \rangle &= \sum_{l=0}^{N(p-1)} \langle (\iota_-^\top (C \otimes I) \iota_-) f_{p-1}^l, C' f_{p-1}^l \rangle \\
&= \frac{p\lambda_{2(p-1)}}{\lambda_{2p}\lambda_{p-1}} \sum_{i=0}^{m} \sum_{l=0}^{N(p-1)} \langle \partial_i C \delta_i f_{p-1}^l, C' f_{p-1}^l \rangle \\
&= \frac{p\lambda_{2(p-1)}}{\lambda_{2p}\lambda_{p-1}} \sum_{i=0}^{m} \sum_{j=0}^{N(p)} \sum_{l=0}^{N(p-1)} \langle \delta_i f_{p-1}^l, f_p^j \rangle \langle \partial_i C f_p^j, C' f_{p-1}^l \rangle \\
&= \frac{2}{\lambda_{2p}} \sum_{i=0}^{m} \sum_{j=0}^{N(p)} \sum_{l=0}^{N(p-1)} \langle f_{p-1}^l, \partial_i f_p^j \rangle \langle \partial_i C f_p^j, C' f_{p-1}^l \rangle \\
&= \frac{p^2}{\lambda_{2p}} \frac{\lambda_{2(p-1)}}{\lambda_{p-1}} \sum_{j=0}^{N(p)} \langle \iota_+(C f_p^j), (C' \otimes I) \iota_+(f_p^j) \rangle \\
&= \frac{p^2}{\lambda_{2p}} \frac{\lambda_{2(p-1)}}{\lambda_{p-1}} \langle C, \Phi_{p-1}^+(C') \rangle.
\end{aligned}
$$

(2.6.2) follows.

Remark. (2.6.3) can be generalized to the effect that

$$D : \mathcal{H}^p \otimes \mathcal{H}^q \to \mathcal{H}^{p-1} \otimes \mathcal{H}^{q-1}$$

corresponds to the map

$$C \mapsto \mu_{p-1} \sum_{i=0}^{m} \partial_i C \delta_i,$$

where $C \in \mathcal{H}^p \otimes \mathcal{H}^q$ is considered as a linear map $C : \mathcal{H}^p \to \mathcal{H}^q$ under the isomorphism $(\mathcal{H}^p)^* = \mathcal{H}^p$ (and so is $D(C)$ under $(\mathcal{H}^{p-1})^* = \mathcal{H}^{p-1}$).

PROOF OF THEOREM 2.6.1. According to Corollary A.3.3 in Appendix 3, the operator D is onto. By (2.6.1), the second statement of Theorem 2.6.1 follows. Since D is onto, its transpose $D^\top$ is injective. By (2.6.2), up to a constant multiple, $D^\top$ is Φ^+. Thus Φ^+ is also injective. The first statement of Theorem 2.6.1 also follows.

Remark. Surjectivity of D plays a key role in the proof of Theorem 2.6.1. In Section 3.5 we will give a different proof of this.

2.7 Quadratic Eigenmaps in Domain Dimension Three

Theorem 2.5.1 implies that the moduli space $\mathcal{L}^p_m$ parametrizing the congruence classes of full p-eigenmaps $f : S^m \to S_V$ is nontrivial iff $m \geq 3$ and $p \geq 2$. The main purpose of this section is to describe the 10-dimensional moduli space $\mathcal{L}^2_3$ corresponding to the first nonrigid range.

The $SU(2)$-equivariant p-eigenmaps will play an important role within $\mathcal{L}^p_3$, $p \geq 2$.

Recall that a map $f : S^3 \to S_V$ is said to be $SU(2)$-*equivariant* if there exists a homomorphism $\rho_f : SU(2) \to SO(V)$ such that

$$f \circ L_g = \rho_f(g) \circ f, \quad g \in SU(2), \tag{2.7.1}$$

where L_g is left multiplication on S^3 by g (as a quaternion). ρ_f defines an $SU(2)$-module structure on V. Under the isomorphisms $V \cong V^* \cong V_f \subset \mathcal{H}^p_3$, V becomes an $SU(2)$-submodule of $\mathcal{H}^p_3|_{SU(2)}$ and the image of f in V is an $SU(2)$-orbit. On the moduli space $\mathcal{L}^p_3$, $SU(2)$-equivariance of f means that the corresponding point $\langle f \rangle \in \mathcal{L}^p_3$ is left fixed by $SU(2)$ (Section 2.3). Thus, the congruence classes of full $SU(2)$-equivariant p-eigenmaps $f : S^3 \to S_V$ are parametrized by the *equivariant moduli space* $(\mathcal{L}^p_3)^{SU(2)}$, the $SU(2)$-fixed points on $\mathcal{L}^p_3$. Since the linear span of $\mathcal{L}^p_3$ is $\mathcal{E}^p_3$, the equivariant moduli space is the linear slice

$$(\mathcal{L}^p_3)^{SU(2)} = \mathcal{L}^p_3 \cap (\mathcal{E}^p_3)^{SU(2)}.$$

We identify $\mathbf{C}^2$ and $\mathbf{R}^4$ in the usual way. ($\mathbf{C}^2 \ni (z,w) = (x+iy, u+iv) \mapsto (x,y,u,v) \in \mathbf{R}^4$.) With this identification, $SU(2)$ becomes a subgroup of $SO(4)$. The orthogonal matrix $\gamma = \operatorname{diag}(1,1,1,-1) \in O(4)$ (or, using complex coordinates $(z,w) \in \mathbf{C}^2$, $\gamma : z \mapsto z$, $w \mapsto \bar{w}$) conjugates $SU(2)$ to the subgroup

$$SU(2)' = \gamma SU(2)\gamma \subset SO(4), \quad \gamma^{-1} = \gamma, \tag{2.7.2}$$

of $SO(4)$ and (as simple computation shows), we have

$$SU(2) \cap SU(2)' = \{\pm I\}.$$

For reasons of dimension

$$SU(2) \cdot SU(2)' = SO(4). \tag{2.7.3}$$

The local product structure (2.7.3) indicates that both $SU(2)$ and $SU(2)'$ are *normal* in $SO(4)$. In particular, $(\mathcal{E}_3^p)^{SU(2)}$ is $SU(2)'$-invariant. Since $-I$ acts on $\mathcal{E}_3^p \subset S^2(\mathcal{H}_3^p)$ as the identity, $(\mathcal{E}_3^p)^{SU(2)}$ is also an $SO(4)$-submodule of $\mathcal{E}_3^p$.

Given an $SU(2)$-equivariant map $f : S^3 \to S_V$, the composition $f \circ \gamma : S^3 \to S_V$ is $SU(2)'$-equivariant in the obvious sense, i.e. (2.7.1) holds with f replaced by $f \circ \gamma$ and $SU(2)$ replaced by $SU(2)'$. The congruence classes of full $SU(2)'$-equivariant p-eigenmaps are parametrized by the equivariant moduli space $(\mathcal{L}_3^p)^{SU(2)'}$, the $SU(2)'$-fixed points on $\mathcal{L}_3^p$. As before, we have

$$(\mathcal{L}_3^p)^{SU(2)'} = \mathcal{L}_3^p \cap (\mathcal{E}_3^p)^{SU(2)'}$$

and $(\mathcal{E}_3^p)^{SU(2)'}$ is $SU(2)$-invariant, thereby an $SO(4)$-submodule of $\mathcal{E}_3^p$.

Lemma 2.7.1. *$(\mathcal{E}_3^p)^{SU(2)}$ and $(\mathcal{E}_3^p)^{SU(2)'}$ are orthogonal in $\mathcal{E}_3^p$.*

PROOF. If $(\mathcal{E}_3^p)^{SU(2)}$ and $(\mathcal{E}_3^p)^{SU(2)'}$ were to intersect nontrivially, then, by (2.7.3), a nonzero intersection would be left fixed by $SO(4)$. This contradicts Corollary 2.3.4. Orthogonality now follows from the fact that both $(\mathcal{E}_3^p)^{SU(2)}$ and $(\mathcal{E}_3^p)^{SU(2)'}$ are $SO(4)$-submodules of $\mathcal{E}_3^p$, and that $S^2(\mathcal{H}_3^p)$ has multiplicity 1 decomposition into irreducible $SO(4)$-modules (Corollary A.3.4 in Appendix 3).

Restricting from $SO(4)$ to $U(2)$, the $SO(4)$-module $\mathcal{H}_3^p$ of *complex* spherical harmonics on S^3 of order p splits as

$$\mathcal{H}_3^p|_{U(2)} = \sum_{c+d=p;\, c,d\geq 0} \mathcal{H}^{c,d}, \tag{2.7.4}$$

where $\mathcal{H}^{c,d}$ is the complex irreducible $U(2)$-module of harmonic polynomials of degree c in z, w and degree d in $\bar{z}, \bar{w}$. (This can be seen by writing a harmonic p-homogeneous polynomial in terms of the variables $z, \bar{z}, w, \bar{w}$.) The center

$$\Gamma = \{\operatorname{diag}(e^{i\theta}, e^{i\theta}) \mid \theta \in \mathbf{R}\} \subset SU(2)' \tag{2.7.5}$$

of $U(2)$ acts on each $\mathcal{H}^{c,d}$ as a character.

Remark. For $p = 2$, (2.7.4) reduces to

$$\mathcal{H}_3^2|U(2) = \mathcal{H}^{2,0} \oplus \mathcal{H}^{1,1} \oplus \mathcal{H}^{0,2}.$$

The complex Veronese map $Ver^{\mathbf{C}} : S^3 \to S^5$ and the Hopf map $Hopf : S^3 \to S^2$ introduced in Examples 1.4.1-1.4.2 are orbit maps in the $U(2)$-modules $\mathcal{H}^{2,0}$ and $\mathcal{H}^{1,1}$. They are obtained by the same recipe as the standard minimal immersion f_p; relative to orthonormal bases in $\mathcal{H}^{2,0}$ and $\mathcal{H}^{1,1}$, the components of $Ver^{\mathbf{C}}$ and $Hopf$ are orthogonal with the same norm.

Restricting (2.7.4) further to $SU(2) \subset U(2)$, we obtain

$$\mathcal{H}_3^p|_{SU(2)} = (p+1)W_p$$

as *complex* $SU(2)$-modules. Here, as in Section 1.4, W_p denotes the (unique) complex irreducible $SU(2)$-module of dimension $p+1$. Actually, the local product structure (2.7.3) gives us

$$\mathcal{H}_3^p = W_p \otimes W_p',$$

where W_p' is the complex $SU(2)'$-module obtained from the $SU(2)$-module W_p by conjugating $SU(2)'$ to $SU(2)$ by γ in $SO(4)$. (For some facts on $SU(2)$-representations used here, see Fulton-Harris [1], Vilenkin [1], or Weingart [1].) More generally, if W is an $SU(2)$-module then W' denotes the $SU(2)'$-module obtained from W by conjugating $SU(2)'$ to $SU(2)$. If, in addition, $-I$ acts on W' trivially then W' becomes an $SO(4)$-module with $SU(2)$ acting trivially on W'. The situation is similar when the roles of $SU(2)$ and $SU(2)'$ are switched.

For $p = 2d$ even, with obvious notations, we have

$$\mathcal{H}_3^{2d} = R_{2d} \otimes R_{2d}', \tag{2.7.6}$$

as *real* modules, where $R_{2d} \subset W_{2d}$ is the real $SU(2)$-submodule defined in Section 1.4. In particular,

$$\mathcal{H}_3^{2d}|_{SU(2)} = (2d+1)R_{2d}. \tag{2.7.7}$$

We showed in Section 1.4 that, up to congruence, a full $SU(2)$-equivariant quadratic eigenmap $f_\xi : S^3 \to S_{R_2} = S^2$ is given by (1.4.9)-(1.4.10). The pairs of coefficients (c_1, c_2) satisfying (1.4.9) can be parametrized by S^2, setting

$$c_1 = i\sin(t) \text{ and } c_2 = \bar{c}_0 = \cos(t) \cdot e^{is}/2, \ |t| \leq \pi/2, s \in \mathbf{R}.$$

Here s and t are viewed as spherical coordinates on S^2. ($t = 0$ gives the equator parametrized by s, and t is the parameter for the parallels of latitude.) Taking congruence classes, this parametrization gives rise to a smooth map of S^2 into $\mathcal{L}_3^2$. It is clear that the parameter values (t, s) and

$(-t, s+\pi)$, corresponding to antipodal points in S^2, give congruent eigenmaps. Looking at the components of f_ξ in (1.4.10), we see that the converse is also true. Thus the mapping of S^2 into $\mathcal{L}_3^2$ factors through the antipodal map of S^2 and gives a smooth imbedding of the real projective plane $\mathbf{R}P^2$ into $\mathcal{L}_3^2$. Let $\mathbf{P}$ denote the image of this imbedding. Summarizing, we see that the set of points in $\mathcal{L}_3^2$ that parametrize the full $SU(2)$-equivariant quadratic eigenmaps $f : S^3 \to S^2$ is a *smooth imbedded real projective plane* $\mathbf{P}$ in $\partial(\mathcal{L}_3^2)^{SU(2)}$.

Recall again from Section 1.4 that both $Ver^{\mathbf{C}}$ and *Hopf* are $SU(2)$-equivariant. In fact, they are also equivariant with respect to the unitary group $U(2) \subset SO(4)$ since $U(2)$ is generated by $SU(2)$ and its center Γ given by (2.7.5) above. According to Example 2.3.15, $Ver^{\mathbf{C}}$ and *Hopf* are also antipodal.

By $U(2)$-equivariance, the isotropy subgroup of $\langle Hopf \rangle$ contains $U(2)$. Since there are no connected closed subgroups between $U(2)$ and $SO(4)$, we have

$$SO(4)^0_{Hopf} = U(2),$$

where the superscript indicates the identity component. In fact, simple computation, in the use of the complex form of the Hopf map (Example 1.4.2), shows that $\gamma g \gamma \in SU(2)'$ is in the isotropy subgroup $SO(4)_{\langle Hopf \rangle}$ iff $g \in SU(2)$ is *diagonal or antidiagonal.* The diagonal elements correspond to Γ and the antidiagonals fill another component, a topological circle. We obtain that $SO(4)/U(2)$ is a 2-fold cover of the $SO(4)$-orbit of $\langle Hopf \rangle$. It is well-known that $SO(4)/U(2) = S^2$ and that the only (smooth) factor of S^2 is $\mathbf{R}P^2$. We conclude that the $SO(4)$-orbit of $\langle Hopf \rangle$ is a smooth imbedded real projective plane. Since it is contained in $\mathbf{P}$ it must *coincide* with $\mathbf{P}$. We obtain that

$$SO(4)(\langle Hopf \rangle) = SU(2)'(\langle Hopf \rangle) = \mathbf{P}.$$

Summarizing, we have the following rigidity result:

Corollary 2.7.2. *Given a full $SU(2)$-equivariant quadratic eigenmap $f : S^3 \to S^2$, we have $f = U \circ Hopf \circ g$, for some $U \in O(3)$ and $g \in SU(2)'$.*

$\mathbf{P}$ is contained in a sphere in $(\mathcal{L}_3^2)^{SU(2)}$ of radius $3\sqrt{2}$ (and center at the origin) since $|\langle Hopf \rangle| = 3\sqrt{2}$ (Example 2.3.15). Since the real projective plane cannot be imbedded into $\mathbf{R}^3$ (or S^3), we obtain that $(\mathcal{L}_3^2)^{SU(2)}$ is at least $4+1 = 5$-dimensional. In a similar vein, $(\mathcal{L}_3^2)^{SU(2)'}$ is also at least 5-dimensional. Thus, $\dim(\mathcal{E}_3^2)^{SU(2)} = \dim(\mathcal{E}_3^2)^{SU(2)'} \geq 5$, so that, by Lemma 2.7.1, $(\mathcal{E}_3^2)^{SU(2)} \oplus (\mathcal{E}_3^2)^{SU(2)'}$ is at least 10-dimensional in $\mathcal{E}_3^2$. On the other hand, we know from Theorem 2.5.1 that $\dim \mathcal{L}_3^2 = \dim \mathcal{E}_3^2 = 10$. Thus, $\dim(\mathcal{E}_3^2)^{SU(2)} = \dim(\mathcal{E}_3^2)^{SU(2)'} = 5$, and we have

$$\mathcal{E}_3^2 = (\mathcal{E}_3^2)^{SU(2)} \oplus (\mathcal{E}_3^2)^{SU(2)'} \tag{2.7.8}$$

as $SO(4)$-modules. Since it is 5-dimensional, it is also clear that $(\mathcal{E}_3^2)^{SU(2)}$ is irreducible as an $SU(2)'$-module. Similarly, $(\mathcal{E}_3^2)^{SU(2)'}$ is irreducible as an $SU(2)$-module. Hence

$$(\mathcal{E}_3^2)^{SU(2)} \cong R_4' \text{ and } (\mathcal{E}_3^2)^{SU(2)'} \cong R_4. \tag{2.7.9}$$

(These will also follow from a more general setting in Section 3.6.) Γ acts diagonally on the orthonormal basis of W_4' obtained from (1.4.5) by replacing w with $\bar{w}$. The fixed points of Γ fill the line $\mathbf{R} \cdot z^2\bar{w}^2$. In fact, looking at the standard action of $SU(2)$ on $\mathbf{C}^2$, we see that the isotropy subgroup of this line consists of those $\gamma g \gamma \in SU(2)'$ for which $g \in SU(2)$ is diagonal or antidiagonal. As before, the $SU(2)'$-orbit of $z^2\bar{w}^2$ is topologically a real projective plane. (Under the first isomorphism in (2.7.9), the lines $\mathbf{R} \cdot \langle Hopf \rangle$ and $\mathbf{R} \cdot z^2\bar{w}^2$ correspond to each other.) Every closed connected 1-parameter subgroup of $SU(2)'$ is conjugate to Γ, and thereby serves as the isotropy subgroup of a point on the orbit $SU(2)'(z^2\bar{w}^2)$. We see that, away from the line $\mathbf{R} \cdot z^2\bar{w}^2$, all $SU(2)'$-orbits are three-dimensional.

Restricting the action to the four-dimensional sphere in R_4' that contains the orbit of $z^2\bar{w}^2$, the $SU(2)'$-orbits thus form a homogeneous family of hypersurfaces with two antipodal singular orbits, that are imbedded real projective planes. (In fact, the orbits are the level hypersurfaces of an essentially unique isoparametric function, a cubic polynomial restricted to the 4-sphere in R_4'. The cubic polynomial is explicitly known: ξ for $\nu = 1$ in Problem 2.28 (b), see also É. Cartan [1-2]. It also follows that the "middle" hypersurface is minimal and self-antipodal, while the the rest of the hypersurfaces are paired in antipodal pairs.) It is a classical result that the singular orbits are minimal. By Corollary 2.5.2, up to congruence, the singular orbits are imbedded as the Veronese surface in S^4. Passing to $(\mathcal{E}_3^2)^{SU(2)}$ by the first isomorphism in (2.7.9), we conclude that $\mathbf{P} \subset (\mathcal{E}_3^2)^{SU(2)}$ *is imbedded minimally into its respective 4-sphere of radius* $3\sqrt{2}$ *as a (scaled) Veronese surface.*

Let $f : S^3 \to S_V$ be a full $SU(2)$-equivariant quadratic eigenmap. As noted above, the space of components V_f is an $SU(2)$-submodule of $\mathcal{H}_3^2|_{SU(2)}$. By (2.7.7), the latter splits as

$$\mathcal{H}_3^2|_{SU(2)} = 3R_2$$

so that

$$\dim V = \dim V_f = 3, 6, 9.$$

Thus the possible spherical range dimensions $\dim S_V$ of a full $SU(2)$-equivariant quadratic eigenmap $f : S^3 \to S_V$ *of boundary type* are 2 and 5.

We now determine $(\mathcal{L}_{Ver^{\mathbf{C}}})^{SU(2)}$. Since Γ fixes $\langle Ver^{\mathbf{C}} \rangle$, it has to leave $(\mathcal{L}_{Ver^{\mathbf{C}}})^{SU(2)}$ invariant. The circle group Γ acts on $\partial(\mathcal{L}_{Ver^{\mathbf{C}}})^{SU(2)}$ and thereby on its linear span $\mathcal{E}_{Ver^{\mathbf{C}}}$ without fixed points (except the origin

in $\mathcal{E}_{Ver^{\mathbf{C}}}$) by the local uniqueness of $U(2)$-fixed points on $\partial(\mathcal{L}_3^2)^{SU(2)}$. Thus $(\mathcal{L}_{Ver^{\mathbf{C}}})^{SU(2)}$ is *even dimensional*. This dimension cannot be four since the boundary of $(\mathcal{L}_{Ver^{\mathbf{C}}})^{SU(2)}$ is contained in $\mathbf{P}$.

$Ver^{\mathbf{C}}$ is linearly nonrigid among $SU(2)$-equivariant quadratic eigenmaps, since each quadratic eigenmap in the one-parameter family $Hopf_\alpha : S^3 \to S^2$, $\alpha \in \mathbf{R}$, given by

$$Hopf_\alpha(z,w) = (e^{2i\alpha}z^2 + \bar{w}^2, 2\Im(e^{i\alpha}zw)).$$

is derived from $Ver^{\mathbf{C}}$. (To see this, compare the components of $Hopf_\alpha$ with those of $Ver^{\mathbf{C}}$ given in Example 1.4.1.) Thus $(\mathcal{L}_{Ver^{\mathbf{C}}})^{SU(2)}$ is two-dimensional. Notice that $Hopf_{\alpha_1} \cong Hopf_{\alpha_2}$ iff $\alpha_1 \equiv \alpha_2 \,(\mathrm{mod}\,\pi)$ so that the corresponding points $\{\langle Hopf_\alpha\rangle\}_{\alpha\in\mathbf{R}/\pi\mathbf{Z}}$ give the entire boundary of $(\mathcal{L}_{Ver^{\mathbf{C}}})^{SU(2)}$. The center Γ rotates $(\mathcal{L}_{Ver^{\mathbf{C}}})^{SU(2)}$ since

$$\mathrm{diag}(e^{i\theta}, e^{i\theta}) \cdot \langle Hopf_\alpha\rangle = \langle Hopf_{\alpha+2\theta}\rangle.$$

It follows that $(\mathcal{L}_{Ver^{\mathbf{C}}})^{SU(2)}$ is a flat two-dimensional disk $\mathcal{D}$. (For an explicit computation, see Problem 2.16. Note also that, as we will see later (Corollary 2.7.7), $(\mathcal{L}_{Ver^{\mathbf{C}}})^{SU(2)}$ coincides with $\mathcal{L}_{Ver^{\mathbf{C}}}$.)

Corollary 2.3.20 applied to $Ver^{\mathbf{C}} : S^3 \to S^5$ says that the cone $\mathcal{C}_{Ver^{\mathbf{C}}}$ with base disk $\mathcal{D}$ and vertex $\langle Hopf\rangle = \langle (Ver^{\mathbf{C}})^o\rangle$ is straight and $\partial\mathcal{C}_{Ver^{\mathbf{C}}}$ is part of the 4-dimensional boundary of $(\mathcal{L}_3^2)^{SU(2)}$.

Consider the interior of $\mathcal{D}$. Through any point of $\mathrm{int}\,\mathcal{D}$ the $SU(2)'$-orbit is transversal to $\mathrm{int}\,\mathcal{D}$. This is an easy application of Theorem 2.3.8. Indeed, $SU(2)'$ has no two-dimensional subgroups, and $SU(2)'$ cannot leave $\mathrm{int}\,\mathcal{D}$ invariant since the centroid of an(y) orbit would then be $SO(4)$-fixed; a contradiction to Corollary 2.3.4. Thus the orbit $SU(2)'(\mathrm{int}\,\mathcal{D})$ is an *open submanifold* in $\partial(\mathcal{L}_3^2)^{SU(2)}$. The boundary points of this orbit are on $\mathbf{P}$ since they correspond to range dimension 2. On the other hand, $\mathbf{P}$ is of codimension 2 in $\partial(\mathcal{L}_3^2)^{SU(2)}$ so that the points corresponding to quadratic eigenmaps with range dimension 5 form a *connected set* in $(\mathcal{L}_3^2)^{SU(2)}$. Since $SU(2)'(\mathrm{int}\,\mathcal{D})$ is a component it must coincide with this set. We obtain

$$SO(4)(\mathcal{D}) = SU(2)'(\mathcal{D}) = \partial(\mathcal{L}_3^2)^{SU(2)}. \tag{2.7.10}$$

As a byproduct, we have the following:

Corollary 2.7.3. *Given a full $SU(2)$-equivariant p-eigenmap $f : S^3 \to S^5$, we have $f \circ g \rightleftharpoons Ver^{\mathbf{C}}$ for some $g \in SU(2)'$.*

Since the points on $\mathbf{P}$ correspond to linearly rigid eigenmaps, Lemma 2.3.6 along with the Krein-Milman theorem (Theorem A.1.1 in Appendix 1) give:

Corollary 2.7.4. *The equivariant moduli $(\mathcal{L}_3^2)^{SU(2)}$ that parametrizes the full $SU(2)$-equivariant quadratic eigenmaps $f : S^3 \to S_V$ is isometric with the convex hull of the real projective plane imbedded minimally in S^4 (of radius $3\sqrt{2}$) as a scaled Veronese surface.*

A very transparent picture of $(\mathcal{L}_3^2)^{SU(2)}$ emerges as follows. The outermost $SU(2)'$-orbit on the boundary is the $SU(2)'$-orbit of the Hopf map, and it is an imbedded Veronese surface in the 4-sphere of $(\mathcal{L}_3^2)^{SU(2)}$ of radius $3\sqrt{2}$. The innermost $SU(2)'$-orbit on the boundary is the $SU(2)'$-orbit of the complex Veronese map, and it is another imbedded Veronese surface in the 4-sphere of $(\mathcal{L}_3^2)^{SU(2)}$ of radius $3/\sqrt{2}$. These two $SU(2)'$-orbits are antipodal and scaled by distortion 2. The points on the innermost orbit are centers of flat 2-disks whose boundary circles are contained in the outermost orbit. By (2.7.10), these 2-disks form a single $SU(2)'$-orbit. (Various 3-dimensional projections of the configuration of the two orbits are depicted in the front page illustration. The Veronese surfaces in S^4 are first stereographically projected to $\mathbf{R}^4$ and then projected to $\mathbf{R}^3$ as Roman surfaces. Notice that only four disks in (2.7.10) are projected *isometrically* to $\mathbf{R}^3$.)

Theorem 2.3.7 asserts that the moduli space $\mathcal{L}_3^2$ is the convex hull of points that correspond to linearly rigid eigenmaps. The full linearly rigid $SU(2)$-equivariant quadratic eigenmaps correspond to the orbit $\mathbf{P}$ in $(\mathcal{L}_3^2)^{SU(2)}$. The same holds for $\mathbf{P}'$ in $(\mathcal{L}_3^2)^{SU(2)'}$. Our next result gives a complete geometric description of the moduli space $\mathcal{L}_3^2$:

Theorem 2.7.5. *$\mathcal{L}_3^2$ is the convex hull of $(\mathcal{L}_3^2)^{SU(2)}$ and $(\mathcal{L}_3^2)^{SU(2)'}$. Hence $\mathcal{L}_3^2$ is the convex hull of two orthogonal real projective spaces $\mathbf{P}$ and $\mathbf{P}'$ imbedded minimally in their respective 4-spheres of radius $3\sqrt{2}$.*

Corollary 2.7.6. *The inradius and circumradius of $\mathcal{L}_3^2$ are*

$$r(\mathcal{L}_3^2) = \frac{3}{2} \quad \text{and} \quad R(\mathcal{L}_3^2) = 3\sqrt{2}.$$

Corollary 2.7.7. *We have*

$$(\mathcal{L}_{Ver^{\mathbf{C}}})^{SU(2)} = \mathcal{L}_{Ver^{\mathbf{C}}}.$$

Before proving Theorem 2.7.5, we will digress from the main line and prove a result that enables us to compute the dimension of the moduli $(\mathcal{L}_f)^{SU(2)}$ for *any* full $SU(2)$-equivariant p-eigenmap $f : S^3 \to S_V$ for p even. Given f, $V \cong V_f \subset \mathcal{H}_3^p|_{SU(2)}$ is an $SU(2)$-submodule, and, by (2.7.7), $V = kR_p$ for some $k = 1, \ldots, p+1$. Using (2.3.7), we obtain

$$\mathcal{E}_f^{SU(2)} \cong (S^2(V_f) \cap \mathcal{E}_3^p)^{SU(2)} = (S_0^2(V_f) \cap \mathcal{E}_3^p)^{SU(2)}. \tag{2.7.11}$$

Here S_0^2 denotes the traceless part of the symmetric square, and the second equality is because $\mathcal{E}^p \subset S_0^2(\mathcal{H}^p)$ (Lemma 2.3.2). On the other hand, by Theorem 2.5.1, (2.1.14) and (2.7.7), we have

$$\begin{aligned}(S_0^2(\mathcal{H}_3^p)/\mathcal{E}_3^p)^{SU(2)} &= (\mathcal{H}_3^2 \oplus \mathcal{H}_2^4 \oplus \ldots \oplus \mathcal{H}_3^{2p})^{SU(2)} \\ &= (3R_2 \oplus 5R_4 \oplus \ldots \oplus (2p+1)R_{2p})^{SU(2)} = 0.\end{aligned}$$

Continuing the computation in (2.7.11), since

$$S_0^2(V_f) \subset S_0^2(\mathcal{H}_3^p) = \mathcal{E}_3^p \oplus S_0^2(\mathcal{H}_3^p)/\mathcal{E}_3^p,$$

we have

$$\mathcal{E}_f^{SU(2)} \cong (S_0^2(V_f) \cap \mathcal{E}_3^p)^{SU(2)} = S_0^2(V_f)^{SU(2)}.$$

To decompose this into irreducible $SU(2)$-modules we need to recall the Clebsch-Gordan decomposition for the tensor product

$$W_a \otimes W_b = \sum_{j=0}^{b} W_{a+b-2j}, \quad a \geq b \geq 0.$$

(See Fulton-Harris [1] or Vilenkin [1].) For a, b even, we obtain

$$R_a \otimes R_b = \sum_{j=0}^{b} R_{a+b-2j}, \quad a \geq b \geq 0,$$

as real $SU(2)$-modules. For $a = b$ even, retaining only the components in $R_a \otimes R_a$ that contribute to the symmetric square, we have

$$S^2(R_a) = \sum_{j=0}^{a/2} R_{2a-4j}.$$

Since $V = V_f = kR_p$, we have

$$\begin{aligned} S^2(V_f)^{SU(2)} &= S^2(kR_p)^{SU(2)} \\ &= kS^2(R_p)^{SU(2)} \oplus \frac{k(k-1)}{2}(R_p \otimes R_p)^{SU(2)} \\ &= kR_0 \oplus \frac{k(k-1)}{2} R_0 \\ &= \frac{k(k+1)}{2} R_0. \end{aligned}$$

Summarizing, we obtain the following:

Theorem 2.7.8. *Let p be even, and $f : S^3 \to S_V$ a full $SU(2)$-equivariant p-eigenmap with $V \cong kR_p$. Then we have*

$$\dim(\mathcal{L}_f)^{SU(2)} = \frac{k(k+1)}{2} - 1.$$

In particular, we have

$$\dim(\mathcal{L}_{Hopf})^{SU(2)} = 0 \quad \text{and} \quad \dim(\mathcal{L}_{Ver^c})^{SU(2)} = 2.$$

Theorem 2.7.8 thus gives alternative proofs for some of the conclusions we made above.

PROOF OF THEOREM 2.7.5. We need to show that any line segment connecting $\partial(\mathcal{L}_3^2)^{SU(2)}$ and $\partial(\mathcal{L}_3^2)^{SU(2)'}$ is entirely contained in $\partial\mathcal{L}_3^2$, since then the union of these line segments make up the entire boundary of $\mathcal{L}_3^2$.

Let $f_1 : S^3 \to S_{V_1}$ and $f_2 : S^3 \to S_{V_2}$ be full quadratic eigenmaps with f_1 $SU(2)$-equivariant and f_2 $SU(2)'$-equivariant. Consider a full quadratic

eigenmap $f : S^3 \to S_V$ whose corresponding point $\langle f \rangle$ is in the interior of the line segment connecting $\langle f_1 \rangle$ and $\langle f_2 \rangle$. The space of components V_{f_1} is an $SU(2)$-invariant proper submodule of $\mathcal{H}^2_3|_{SU(2)}$, and V_{f_2} is an $SU(2)'$-invariant proper submodule of $\mathcal{H}^2_3|_{SU(2)'}$. The following lemma generalizes this situation.

Lemma 2.7.9. *Let G be a compact Lie group, R an absolutely irreducible (real) G-module, and W a trivial G-module. Then any G-submodule Z of $R \otimes W$ is of the form $Z = R \otimes W_0$, where $W_0 \subset W$ is a linear subspace.*

PROOF. Without loss of generality, Z can be assumed to be irreducible. Let $\{w_s\}_{s=1}^n \subset W$ be a basis and write $R \otimes W = \sum_{s=1}^n R_s$, as G-modules, where $R_s = R \otimes \mathbf{R} \cdot w_s \cong R$, $s = 1, \ldots, n$. Now the statement follows from Schur's lemma applied to the projections $Z \to R_s$.

Returning to our proof of Theorem 2.7.5, we have

$$V_{f_1} = R_2 \otimes W_0' \quad \text{and} \quad V_{f_2} = W_0 \otimes R_2',$$

where $W_0 \subset R_2$ and $W_0' \subset R_2'$ are linear subspaces. On the other hand, by Theorem 2.3.5, we have

$$V_f = V_{f_1} + V_{f_2} = R_2 \otimes W_0' + W_0 \otimes R_2'. \tag{2.7.12}$$

Since V_{f_1} and V_{f_2} are proper submodules, it follows that V_f is a proper linear subspace of $\mathcal{H}^2_3$. Thus f is of boundary type: $\langle f \rangle \in \partial \mathcal{L}^2_3$. Theorem 2.7.5 follows.

Corollary 2.7.10. *The possible spherical range dimensions of a full quadratic eigenmap $f : S^3 \to S_V$ are*

$$\dim S_V = 2, 4, 5, 6, 7, 8.$$

In particular, there is no full quadratic eigenmap $f : S^3 \to S^3$.

PROOF. By (2.7.12), we have

$$\begin{aligned} \dim V &= \dim V_f \\ &= \dim R_2 \otimes W_0' + \dim W_0 \otimes R_2' - \dim W_0 \otimes W_0' \\ &= 3(\dim W_0 + \dim W_0') - \dim W_0 \dim W_0'. \end{aligned}$$

Since $\dim W_0, \dim W_0' \leq 2$, the corollary follows.

Corollary 2.7.11. *A full quadratic eigenmap $f : S^3 \to S_V$ is $SU(2)$-or $SU(2)'$-equivariant iff $\dim V$ is divisible by 3.*

Remark. The role of $SU(2)$-equivariant maps in $\mathcal{L}^p_3$ was first exploited in Toth-Ziller [1] for spherical minimal immersions. For quadratic eigenmaps, the treatment here is new. The main result of this section, Theorem 2.7.5, first appeared in Toth [6]. We present here a different (and much simpler) proof.

2.8 Raising the Domain Dimension

The degree-raising and -lowering operators enable us to manufacture eigenmaps of arbitrary degree from a given eigenmap without changing the domain dimension. In this section, we construct a domain-dimension-raising operator for eigenmaps. Domain-dimension-raising appeared first in Gauchman-Toth [1] to solve the DoCarmo-Wallach problem for spherical minimal immersions with higher order isotropy. This will be discussed in Chapter 3.

In the following, we use the notations introduced in Section 2.1. Since keeping track of the domain dimension is important, we reintroduce here the subscript m for various geometric objects.

To define the domain-dimension-raising operator, we need to take a more analytical look at the decomposition in (2.1.15). It is technically more convenient to raise the domain dimension m by one. We thus consider p-homogeneous polynomials $\xi \in \mathcal{P}^p_{m+2}$ in the variables $x = (x_0, \ldots, x_m) \in \mathbf{R}^{m+1}$ and $x_{m+1} \in \mathbf{R}$ and write $\xi(x, x_{m+1})$ when display of the arguments is necessary.

In view of (2.1.15) (with m replaced by $m+1$), given $\chi \in \mathcal{H}^p_{m+1} \subset \mathcal{P}^p_{m+2}$, we can write

$$\chi(x, x_{m+1}) = \sum_{q=0}^{p} H(x_{m+1}^{p-q} \chi_q(x)), \tag{2.8.1}$$

where $\chi_q \in \mathcal{H}^q_m$, $q = 0, \ldots, p$. Using the harmonic projection formula (2.1.7) for $\xi(x) = x_{m+1}^{p-q}\chi_q(x)$, we can work out each term on the right-hand side. Since χ_q is harmonic and does not depend on the variable x_{m+1}, we have

$$\triangle^j(x_{m+1}^{p-q}\chi_q(x)) = (p-q)(p-q-1)\ldots(p-q-2j+1)x_{m+1}^{p-q-2j}\chi_q(x).$$

Substituting this into (2.1.7) and repeating essentially the same computations as in Section 2.1 leading to (2.1.11), we arrive at the following

$$H(x_{m+1}^{p-q}\chi_q(x)) = \frac{(p-q)!\Gamma(q+m/2)}{2^{p-q}\Gamma(p+m/2)}\rho^{p-q}C^{m/2+q}_{p-q}\left(\frac{x_{m+1}}{\rho}\right)\chi_q(x), \tag{2.8.2}$$

where Γ is the *Gamma function* and C^a_d is the *ultraspherical polynomial* (2.1.10). (Note that (2.8.2) reduces to (2.1.11) for $\chi_q = 1$.)

Observe that the factor

$$K_{m,p,q}(\rho^2, x_{m+1}) = \frac{(p-q)!\Gamma(q+m/2)}{2^{p-q}\Gamma(p+m/2)}\rho^{p-q}C^{m/2+q}_{p-q}\left(\frac{x_{m+1}}{\rho}\right) \tag{2.8.3}$$

in (2.8.2) is a (harmonic) *polynomial* in x_{m+1} and ρ^2.

We need an elementary integral formula that establishes a relation between integrals on $S^{m+1} \subset \mathbf{R}^{m+2}$ and on the subsphere $S^m \subset \mathbf{R}^{m+1}$ (given by $x_{m+1} = 0$).

Lemma 2.8.1. *Given* $\xi \in \mathcal{P}^r_{m+1}$ *and* $\eta : [0,\pi] \to \mathbf{R}$ *any continuous function, we have*

$$\int_{S^{m+1}} \xi(x)\eta(x_{m+1})v_{S^{m+1}} = \int_0^\pi \eta(\cos\phi)\sin^{m+r}\phi\, d\phi \cdot \int_{S^m} \xi(x)v_{S^m}. \tag{2.8.4}$$

PROOF. Consider the transformation

$$\gamma : [0,\pi] \times \mathbf{R}^{m+1} \to \mathbf{R}^{m+2}$$

defined by

$$\gamma(\phi, x) = (\sin\phi \cdot x, \cos\phi \cdot |x|), \quad x = (x_0, \ldots, x_m) \in \mathbf{R}^{m+1},\ \phi \in [0,\pi].$$

Restricting to the unit spheres, γ projects the cylinder $[0,\pi] \times S^m$ to S^{m+1} with 0 and π corresponding to the north and south poles. The determinant of the Jacobian of γ is

$$(-1)^m |x| \sin^m \phi.$$

Now the general change of variable formula specializes to (2.8.4).

Lemma 2.8.2. *With respect to the decomposition (2.8.1), the standard minimal immersion* $f_{m+1,p} : S^{m+1} \to S_{\mathcal{H}^p_{m+1}}$ *can be written as*

$$f_{m+1,p}(x, x_{m+1}) = \left(c_{m,p,q} H(x^{p-q}_{m+1} f_{m,q}(x)) \right)_{0 \le q \le p}, \tag{2.8.5}$$

where $x = (x_0, \ldots, x_m)$ *and*

$$c_{m,p,q} = \binom{p}{q}(m+2q-1)\frac{(m+q-2)!}{(m+p-1)!}\frac{[m(m+2)\ldots(m+2p-2)]^2}{m(m+1)\ldots(m+p+q-1)} \tag{2.8.6}$$

PROOF. For $q = 0, \ldots, p$, we fix orthonormal bases $\{f^{j_q}_{m,q}\}_{j_q=0}^{N(m,q)} \subset \mathcal{H}^q_m$. Writing $f_{m,q}$ in terms of these as components, we need to show that the spherical harmonics in (2.8.5) are orthonormal. By (2.8.2), up to a constant multiple

$$\int_{S^{m+1}} H(x^{p-q}_{m+1} f^{j_q}_{m,q}(x)) H(x^{p-q'}_{m+1} f^{j'_{q'}}_{m,q'}(x)) v_{S^{m+1}}$$

is equal to

$$\int_{S^{m+1}} f^{j_q}_{m,q}(x) f^{j'_{q'}}_{m,q'}(x) C^{m/2+q}_{p-q}(x_{m+1}) C^{m/2+q'}_{p-q'}(x_{m+1}) v_{S^{m+1}}.$$

By Lemma 2.8.1, this is rewritten as

$$\int_0^\pi C^{m/2+q}_{p-q}(\cos\phi) C^{m/2+q'}_{p-q'}(\cos\phi) \sin^{m+q+q'}\phi\, d\phi \int_{S^m} f^{j_q}_{m,q}(x) f^{j'_{q'}}_{m,q'}(x) v_{S^m}.$$

Now orthogonality follows from (2.1.12) and orthogonality of the components of the standard minimal immersions. It remains to work out the normalization constants $c_{m,p,q}$ in (2.8.6). For brevity, for fixed $q = 0, \ldots, p$, we replace $f^j_{m,q}$ by a spherical harmonic $\chi_q \in \mathcal{H}^q_m$ and assume that

$$|\chi_q|^2 = \frac{N(m,q)+1}{\operatorname{vol}(S^m)} \int_{S^m} \chi_q(x)^2 v_{S^m} = 1. \tag{2.8.7}$$

Using (2.8.2), (2.8.4) and (2.1.13), we compute

$$\begin{aligned}
\int_{S^{m+1}} H(x_{m+1}^{p-q}\chi_q(x))^2 v_{S^{m+1}} &= \left(\frac{(p-q)!\Gamma(q+m/2)}{2^{p-q}\Gamma(p+m/2)}\right)^2 \\
&\times \int_{S^{m+1}} C_{p-q}^{m/2+q}(x_{m+1})^2 \chi_q(x)^2 v_{S^{m+1}} \\
&= \left(\frac{(p-q)!\Gamma(q+m/2)}{2^{p-q}\Gamma(p+m/2)}\right)^2 \\
&\times \int_0^\pi C_{p-q}^{m/2+q}(\cos\phi)^2 \sin^{m+2q}\phi \, d\phi \\
&\times \int_{S^m} \chi_q(x)^2 v_{S^m} \\
&= \frac{\pi(p-q)!\Gamma(p+q+m)}{2^{2p+m-1}(p+m/2)\Gamma(p+m/2)^2} \\
&\times \int_{S^m} \chi_q(x)^2 v_{S^m}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
|H(x_{m+1}^{p-q}\chi_q(x))|^2 &= \frac{N(m+1,p)+1}{\operatorname{vol}(S^{m+1})} \int_{S^{m+1}} H(x_{m+1}^{p-q}\chi_q(x))^2 v_{S^{m+1}} \\
&= \frac{\pi(p-q)!\Gamma(p+q+m)}{2^{2p+m-1}(p+m/2)\Gamma(p+m/2)^2} \\
&\times \frac{N(m+1,p)+1}{N(m,q)+1} \frac{\operatorname{vol}(S^m)}{\operatorname{vol}(S^{m+1})}.
\end{aligned} \tag{2.8.8}$$

For $p = q = 0$, (2.8.8) gives the ratio:

$$\frac{\operatorname{vol}(S^m)}{\operatorname{vol}(S^{m+1})} = \frac{2^{m-2}m\Gamma(m/2)^2}{\pi\Gamma(m)}.$$

Substituting this back into (2.8.8), the value of $c_{m,p,q}$ in (2.8.6) follows.

Remark. In the last step of the proof we obtained the recurrence

$$\operatorname{vol}(S^{m+1}) = \frac{\pi\Gamma(m)}{2^{m-2}m\Gamma(m/2)^2} \operatorname{vol}(S^m), \quad \operatorname{vol}(S^1) = 2\pi.$$

Using the duplication formula

$$\Gamma(2z) = \frac{2^{2z-1}\Gamma(z)\Gamma(z+1/2)}{\sqrt{\pi}},$$

for $z = m/2$, and $\Gamma(z+1) = z\Gamma(z)$, the recurrence formula is rewritten as

$$\frac{\Gamma\left(\frac{m+2}{2}\right)}{2\pi^{\frac{m+2}{2}}}\operatorname{vol}(S^{m+1}) = \frac{\Gamma\left(\frac{m+1}{2}\right)}{2\pi^{\frac{m+1}{2}}}\operatorname{vol}(S^m).$$

Since this is 1 for $m = 1$, we obtain the volume of the m-sphere

$$\operatorname{vol}(S^m) = \frac{2\pi^{\frac{m+1}{2}}}{\Gamma\left(\frac{m+1}{2}\right)}$$

(cf. Section 2.1).

Finally, we are ready to define the domain-dimension-raising operator. We first split off $f_{m,p}$ in (2.8.5) corresponding to $p = q$:

$$f_{m+1,p}(x, x_{m+1}) = \left(c_{m,p,p}f_{m,p}(x), (c_{m,p,q}H(x_{m+1}^{p-q}f_{m,q}(x))_{0\le q\le p-1}\right). \tag{2.8.9}$$

Let $f : S^m \to S_V$ be a full p-eigenmap. We define

$$\tilde{f} : \mathbf{R}^{m+2} \to V \oplus (\mathcal{H}^p_{m+1}/\mathcal{H}^p_m),\ \ \mathcal{H}^p_{m+1}/\mathcal{H}^p_m = \sum_{q=0}^{p-1}\mathcal{H}^q_m,$$

by

$$\tilde{f}(x, x_{m+1}) = \left(c_{m,p,p}f(x), (c_{m,p,q}H(x_{m+1}^{p-q}f_{m,q}(x))_{0\le q\le p-1}\right). \tag{2.8.10}$$

To show that $\tilde{f}$ is spherical we note that $f_{m+1,p} = \tilde{f}_{m,p}$ differs from $\tilde{f}$ by replacing $f_{m,p}$ by f. Thus

$$\begin{aligned}|\tilde{f}|^2 - |f_{m+1,p}|^2 &= |\tilde{f}|^2 - |\tilde{f}_{m,p}|^2\\ &= c^2_{m,p,p}(|f|^2 - |f_{m,p}|^2).\end{aligned}$$

This is zero since f is spherical.

Restricting $\tilde{f}$ to the unit spheres, we define the p-eigenmap

$$\tilde{f} : S^{m+1} \to S_{V\oplus(\mathcal{H}^p_{m+1}/\mathcal{H}^p_m)}.$$

We say that $\tilde{f}$ is obtained from f by *raising the domain dimension.* It is clear that $\tilde{f}$ is full. We then obtain the following:

Theorem 2.8.3 *Given a full p-eigenmap $f : S^m \to S_V$, the p-homogeneous harmonic polynomial map $\tilde{f}$ defined by (2.8.10) is spherical and restricts to a full p-eigenmap $\tilde{f} : S^{m+1} \to S_{V\oplus(\mathcal{H}^p_{m+1}/\mathcal{H}^p_m)}$.*

Raising the domain dimension $f \mapsto \tilde{f}$ gives rise to a map $\Theta_m : \mathcal{L}^p_m \to \mathcal{L}^p_{m+1}$ between the standard moduli spaces. Indeed, let $f = A \circ f_{m,p}$ with $A : \mathcal{H}^p_m \to V$. Comparing (2.8.9) and (2.8.10), we have

$$\tilde{f} = (A \oplus I_{\mathcal{H}^p_{m+1}/\mathcal{H}^p_m}) \circ f_{m+1,p}.$$

Thus

$$\begin{aligned}\Theta_m(\langle f\rangle) = \langle \tilde{f}\rangle &= (A^\top A - I_{\mathcal{H}^p_m}) \oplus 0_{\mathcal{H}^p_{m+1}/\mathcal{H}^p_m} \\ &= \langle f\rangle \oplus 0_{\mathcal{H}^p_{m+1}/\mathcal{H}^p_m}.\end{aligned}$$

Hence Θ_m extends to an $SO(m+1)$-module homomorphism

$$\Theta_m : \mathcal{E}^p_m \to \mathcal{E}^p_{m+1}$$

induced by the inclusion $S^2(\mathcal{H}^p_m) \subset S^2(\mathcal{H}^p_{m+1})$, $C \mapsto C \oplus 0_{\mathcal{H}^p_{m+1}/\mathcal{H}^p_m}$, $C \in S^2(\mathcal{H}^p_m)$. Finally, note that if $f : S^m \to S^n$ is a p-eigenmap with $n < N(m,p)$ then $\tilde{f}$ has range dimension

$$n + N(m+1,p) - N(m,p) (< N(m+1,p)).$$

In particular, Θ_m *carries boundary points of* $\mathcal{L}^p_m$ *to boundary points of* $\mathcal{L}^p_{m+1}$.

Summarizing, Θ_m imbeds $\mathcal{L}^p_m$ into $\mathcal{L}^p_{m+1}$ as a linear slice and the imbedding is equivariant with respect to the inclusion $SO(m+1) \subset SO(m+2)$.

The domain-dimension-raising operator is obtained by replacing the first component in the decomposition (2.8.9) with a full p-eigenmap. A more general operator can be obtained by replacing each component $f_{m,q}$ in (2.8.9) with full q-eigenmaps, $q = 1, \ldots, p$.

Theorem 2.8.4. *Given full q-eigenmaps $g_q : S^m \to S^{n_q}$, $q = 1, \ldots, p$, the map $\tilde{g} : \mathbf{R}^{m+2} \to \mathbf{R}^{N+1}$, $N = \sum_{q=1}^p (n_q + 1)$, defined by*

$$\tilde{g}(x, x_{m+1}) = (c_{m,p,q} H(x^{p-q}_{m+1} g_q(x))_{0 \le q \le p}) \tag{2.8.11}$$

is spherical, and it restricts to a full p-eigenmap $\tilde{g} : S^{m+1} \to S^N$.

PROOF. As noted above, $K_{m,p,q}$ in (2.8.3) is a polynomial in ρ^2 and x_{m+1}. By (2.8.9) and (2.8.10), we have

$$\begin{aligned}|\tilde{g}|^2 - |f_{m+1,p}|^2 &= \sum_{q=0}^p c^2_{m,p,q} (|H(x^{p-q}_{m+1} g_q(x))|^2 - |H(x^{p-q}_{m+1} f_{m,q}(x))|^2) \\ &= \sum_{q=0}^p c^2_{m,p,q} K_{m,p,q}(\rho^2, x_{m+1})^2 (|g_q(x)|^2 - |f_{m,q}(x)|^2) = 0.\end{aligned}$$

Sphericality of (2.8.11) follows.

2.9 Additional Topic: Quadratic Eigenmaps

A. Generalities on Quadratic Eigenmaps

In this section, we develop an independent treatment for quadratic eigenmaps. The advantage lies in gaining geometric insight into some of the general features of eigenmaps without the use of any representation theory.

Let $f : S^m \to S_V$ be a quadratic form. We write f as

$$f(x) = \sum_{r=0}^{m} a_r x_r^2 + \sum_{0 \le s < t \le m} a_{st} x_s x_t, \quad x = (x_0, \ldots, x_m) \in \mathbf{R}^{m+1}, \quad (2.9.1)$$

where $a_r, a_{st} \in V$, $r = 0, \ldots, m$, $0 \le s < t \le m$. We set $a_{st} = a_{ts}$, so that a_{st} is defined for all *distinct* indices $0 \le s \ne t \le m$. Substituting (2.9.1) into the condition of sphericality $|f(x)|^2 = \rho^4$ and expanding, we obtain that f is spherical iff

$$\begin{aligned} |a_r| &= 1, \\ \langle a_r, a_{rs} \rangle &= 0, && r, s \text{ distinct} \\ |a_{rs}|^2 + 2\langle a_r, a_s \rangle &= 2, && r, s \text{ distinct} \\ \langle a_r, a_{st} \rangle + \langle a_{rs}, a_{rt} \rangle &= 0, && r, s, t \text{ distinct} \\ \langle a_{rs}, a_{kl} \rangle + \langle a_{rk}, a_{sl} \rangle + \langle a_{rl}, a_{sk} \rangle &= 0, && r, s, k, l \text{ distinct}. \end{aligned} \quad (2.9.2)$$

If the unit vectors a_r and a_s are linearly independent, then the first three equations of (2.9.2) mean that the three vectors a_r, a_s, a_{rs} are linearly independent, and, in their three-dimensional linear span, a_{rs} is $\sqrt{2}/\sqrt{1 + \langle a_r, a_s \rangle}$ times the vector cross product $a_r \times a_s$ provided that the orientation is chosen properly. This is because

$$|a_r \times a_s| = \sqrt{1 - \langle a_r, a_s \rangle^2} = |a_{rs}| \sqrt{1 + \langle a_r, a_s \rangle} / \sqrt{2}.$$

A system of vectors $\{a_r\}_{r=0}^m \cup \{a_{st}\}_{0 \le s < t \le m} \subset V$, is said to be *feasible* if it satisfies (2.9.2). Given a quadratic form f with associated feasible system, f is a (quadratic) *eigenmap* iff its components are harmonic. Since $\triangle f(x) = 2 \sum_{r=0}^m a_r$, f is an eigenmap iff

$$\sum_{r=0}^{m} a_r = 0. \quad (2.9.3)$$

Given a quadratic form $f : S^m \to S_V$ with associated feasible system of vectors $\{a_r\}_{r=0}^m \cup \{a_{st}\}_{0 \le s < t \le m} \subset V$, for $U \in O(V)$, the congruent quadratic form $U \circ f$ has $\{Ua_r\}_{r=0}^m \cup \{Ua_{st}\}_{0 \le s < t \le m} \subset V$ as the associated feasible system of vectors.

Let $f : S^m \to S_V$ be a full quadratic form with associated feasible system of vectors as above. Let $F : \mathbf{R}^{m+1} \to W$, $W = \operatorname{span} \{a_{st}\}_{0 \le s < t \le m}$, be the

mixed part of f in (2.9.1):

$$F(x) = \sum_{0 \le s < t \le m} a_{st} x_s x_t.$$

We call $F : \mathbf{R}^{m+1} \to W$ *the quadratic polynomial map associated to the quadratic form* f. By the last equation of (2.9.2), we have

$$|F(x)|^2 = \sum_{0 \le s < t \le m} \mu_{st} x_s^2 x_t^2 + \sum_{r=0}^{m} \sum_{0 \le s < t \le m} \nu_{st}^r x_r^2 x_s x_t, \tag{2.9.4}$$

where

$$\mu_{st} = \mu_{ts} = |a_{st}|^2, \quad \nu_{st}^r = \nu_{ts}^r = \langle a_{rs}, a_{rt} \rangle, \quad \nu_{sr}^r = 0.$$

In terms of these, (2.9.3) is rewritten as

$$\sum_{s=0}^{m} \mu_{st} = 2(m+1), \text{ for all } t = 0, \ldots, m$$

and

$$\sum_{r=0}^{m} \nu_{st}^r = 0, \text{ for all } 0 \le s < t \le m.$$

We first claim that F determines f up to congruence. To show this we can consider the decomposition $a_r = a_r^\top + a_r^\perp$ relative to $V = W \oplus W^\perp$, i.e. $a_r^\top \in W$ and $a_r^\perp \in W^\perp$. The second and fourth equations in (2.9.2) give

$$\begin{aligned} \langle a_r^\top, a_{rs} \rangle &= 0, \\ \langle a_r^\top, a_{st} \rangle &= -\nu_{st}^r, \quad r, s, t \text{ distinct.} \end{aligned}$$

Since W is spanned by $\{a_{st}\}_{0 \le s < t \le m}$, unicity of $a_r^\top$ follows. Now, for the entries of the Gram matrix of the orthogonal components $\{a_r^\perp\}$, we have

$$\begin{aligned} |a_r^\perp|^2 &= 1 - |a_r^\top|^2, \\ \langle a_s^\perp, a_t^\perp \rangle &= 1 - \langle a_s^\top, a_t^\top \rangle - \frac{\mu_{st}}{2}. \end{aligned}$$

Since, up to isometry, the Gram matrix determines the corresponding system of vectors (in $W^\perp \subset \mathbf{R}^{n+1}$), unicity of $a_r^\perp$ follows.

B. Separable Eigenmaps

We now introduce the concept of separability which plays an important part in this section not only because of the abundance of classical examples which have this property but because it enables us to visualize the moduli space $\mathcal{L}_3^2$.

Let $\xi \in \mathcal{P}^2$ be a quadratic polynomial in $x_0, \ldots, x_m$. We say that ξ is *pure* (resp. *mixed*) if $\xi(x) = \sum_{r=0}^{m} c_r x_r^2$ (resp. $\xi(x) = \sum_{0 \le s < t \le m} c_{st} x_s x_t$).

A quadratic polynomial map $f : \mathbf{R}^{m+1} \to \mathbf{R}^{n+1}$ (resp. quadratic form $f : S^m \to S^n$) is *separated* if f is congruent to a quadratic polynomial map (resp. quadratic form) such that each component (relative to the standard basis) is pure or mixed. A quadratic polynomial map $f : \mathbf{R}^{m+1} \to \mathbf{R}^{n+1}$ (resp. quadratic form $f : S^m \to S^n$) is *separable* if there exists an isometry $g \in O(m+1)$ such that $f \circ g$ is separated.

In terms of the feasible system of vectors, a quadratic form f is separated iff

$$\langle a_r, a_{st} \rangle = 0, \quad \text{for all } r, s, t = 0, \ldots, m, \ s \neq t$$

or equivalently, iff

$$\nu_{st}^r = 0.$$

If $V = \mathbf{R}^{n+1}$, we will always take $f : S^m \to S^n$ in its congruence class such that $\{a_r\}_{r=0}^m \subset \mathbf{R}^u$, where $\mathbf{R}^u \subset \mathbf{R}^{n+1}$ is the linear subspace spanned by the first u coordinates, and $\{a_{st}\}_{0 \leq s < t \leq m} \subset \mathbf{R}^{v+1}$, where $\mathbf{R}^{v+1} = (\mathbf{R}^u)^\perp$, $u + v = n$.

Example 2.9.1. We give an explicit formula for the standard minimal immersion $f_{m,2} : S^m \to S^N$, where

$$N + 1 = N(m, 2) = m + \binom{m+1}{2} = \frac{m(m+3)}{2}.$$

Since this is still classical, we will also call it the *Veronese map* and denote it by $Ver_m : S^m \to S^N$. Keeping track of the domain dimension and the degree will eliminate any confusion between $Ver_m : S^m \to S^N$ and the generalized Veronese map $Ver_p : S^2 \to S^{2p}$ of degree p introduced in Section 1.4. In any case, for $m = p = 2$ these maps are congruent, as we will see below. We first define $Ver_m : \mathbf{R}^{m+1} \to \mathbf{R}^{N+2}$ by listing its components:

$$\sqrt{1 + \frac{1}{m}} \left(x_r^2 - \frac{\rho^2}{m+1} \right), \ r = 0, \ldots, m, \ \sqrt{2}\sqrt{1 + \frac{1}{m}} x_s x_t, \ 0 \leq s < t \leq m,$$

where as usual we set $\rho^2 = |x|^2 = x_0^2 + \ldots + x_m^2$, $x = (x_0, \ldots, x_m) \in \mathbf{R}^{m+1}$. The first $m+1$ components add up to zero and, this being the only linear relation among the components, the image of Ver_m spans the corresponding linear hyperplane $\mathbf{R}_0^{N+1} \subset \mathbf{R}^{N+2}$. We have

$$\mathbf{R}_0^{N+1} = \mathbf{R}_0^m \times \mathbf{R}^{N+1-m},$$

where

$$\mathbf{R}_0^m = \left\{ (y_0, \ldots, y_m) \in \mathbf{R}^{m+1} \,\middle|\, \sum_{r=0}^m y_r = 0 \right\}.$$

The squares of the components of Ver_m add up to ρ^4 so that, restricting (and retaining the notation), we obtain a full quadratic eigenmap $Ver_m : S^m \to S_0^N = S_{\mathbf{R}_0^{N+1}}$. That this is congruent to the standard minimal

immersion $f_{m,2}$ follows either by checking $SO(m+1)$-equivariance of Ver_m, or by working out orthonormality of the components of Ver_m relative to an orthonormal basis in the hyperplane $\mathbf{R}_0^{N+1} \subset \mathbf{R}^{N+2}$. Up to congruence, the Veronese map $Ver_m : S^m \to S^N$ can be characterized by saying that it is separated, $\langle a_r, a_s \rangle = -1/(m+1)$, $r \neq s$, and $\{a_{st}\}_{0 \leq s < t \leq m}$ is orthogonal with $|a_{st}|^2 = 2(1 + 1/m)$. Geometrically, $a_0, \ldots, a_m \in \mathbf{R}^m$ are the vertices of a regular m-simplex with centroid at the origin, and $\{a_{st}\}_{0 \leq s < t \leq m} \subset \mathbf{R}^{m(m+1)/2}$ is orthonormal with the same norm $\sqrt{2}\sqrt{1 + 1/m}$.

Returning to the main line, let f be separable with associated quadratic polynomial map $F : \mathbf{R}^{m+1} \to W$. Then (2.9.4) reduces to

$$|F(x)|^2 = \sum_{0 \leq s < t \leq m} \mu_{st} x_s^2 x_t^2. \tag{2.9.5}$$

Conversely, let $F : \mathbf{R}^{m+1} \to W$ be a quadratic polynomial map satisfying (2.9.4), and assume that

$$G(\mu) = \left(1 - \frac{\mu_{st}}{2}\right)_{s,t=0}^{m} \geq 0, \quad \mu_{st} = \mu_{ts} \geq 0, \ \mu_{rr} = 0.$$

Then, up to isometry, there exists $\{a_r\}_{r=0}^m \subset \mathbf{R}^u$, $u = \operatorname{rank} G(\mu)$, such that the Gram matrix of the system $\{a_r\}_{r=0}^m$ is $G(\mu)$. Putting a_r and a_{st} (which define F) together in $\mathbf{R}^u \oplus W$, by the results of Section 2.9/A, we obtain a separated form $f : S^m \to S_V$, where $V = \mathbf{R}^u \oplus W$. Finally, f is an eigenmap iff the sum of the entries of each row in $G(\mu)$ is zero, i.e. iff

$$\sum_{s=0}^{m} \mu_{st} = 2(m+1), \text{ for all } t = 0, \ldots, m.$$

Extracting all these facts about the coefficients μ_{st}, we introduce the *signature space*

$$\Sigma_m = \{\mu = (\mu_{st})_{s,t=0}^m \in \mathbf{R}^{m(m+1)/2} \mid G(\mu) = (1 - \mu_{st}/2)_{s,t=0}^m \geq 0, \\ \mu_{st} = \mu_{ts} \geq 0, \ \mu_{rr} = 0\}.$$

Note that Σ_m is convex and semi-algebraic (since it is defined by polynomial inequalities such as nonnegativity of the diagonal minors of $G(\mu)$). By a *quadratic polynomial map F with signature $\mu \in \Sigma_m$* we mean a quadratic polynomial map $F : \mathbf{R}^{m+1} \to W$ satisfying (2.9.5). Summarizing the discussion above, to each full *separated form* $f : S^m \to S^n$ there corresponds a full quadratic polynomial map $F : \mathbf{R}^{m+1} \to \mathbf{R}^{v+1}$ of signature $\mu \in \Sigma_m$. The correspondence $f \mapsto F$ is one-to-one.

The signatures corresponding to quadratic *eigenmaps* comprise the slice:

$$\Sigma_m^0 = \left\{ \mu \in \Sigma_m \;\middle|\; \sum_{s=0}^{m} \mu_{st} = 2(m+1), \text{ for all } t = 0, \ldots, m \right\}.$$

As before, this is a semi-algebraic submanifold of Σ_m.

We now construct an imbedding $\tau : \Sigma^0_{m+1} \to \Sigma_m$ with image

$$\tau(\Sigma^0_{m+1}) = \left\{ \mu \in \Sigma_m \,\middle|\, \sum_{0 \le s < t \le m} \mu_{st} = m(m+2) \right\}, \tag{2.9.6}$$

the intersection of Σ_m and the affine hyperplane defined by

$$\sum_{0 \le s < t \le m} \mu_{st} = m(m+2). \tag{2.9.7}$$

Let $\bar{\mu} \in \Sigma^0_{m+1}$, and define $\tau(\bar{\mu}) = \mu \in \mathbf{R}^{m(m+1)/2}$ as $\bar{\mu} \in \mathbf{R}^{(m+1)(m+2)/2}$ with the components $\bar{\mu}_{s,m+1}$, $s = 0, \ldots, m$, deleted. When $\bar{\mu}$ is considered as a symmetric $(m+2) \times (m+2)$-matrix, the $(m+1) \times (m+1)$-matrix μ is obtained from $\bar{\mu}$ by deleting the last row and last column. Since $\bar{\mu} \in \Sigma^0_m$, the sum of entries of each row of the Gram-matrix $G(\bar{\mu})$ is zero. Thus we have

$$\begin{aligned} \bar{\mu}_{s,m+1} &= 2(m+2) - \sum_{t=0}^{m} \bar{\mu}_{st} \\ &= 2(m+2) - \sum_{t=0}^{m} \mu_{st}. \end{aligned} \tag{2.9.8}$$

In particular, $\det G(\bar{\mu}) = 0$. Hence, $G(\bar{\mu}) \ge 0$ iff $G(\mu) \ge 0$ and $\mu \in \Sigma_m$ follows. Moreover, μ completely determines $\bar{\mu}$ by (2.9.8). Finally, the sum of the entries of the last row of $G(\bar{\mu})$ is zero. By symmetry, these are precisely the entries of the last column given by (2.9.8). Adding up and using symmetry again we arrive at (2.9.7). Since this is the defining relation in (2.9.6) we obtain that $\tau : \bar{\mu} \mapsto \mu$ is a bijection between Σ^0_{m+1} and the slice in (2.9.6).

Given $\mu \in \Sigma_m$ there is a universal quadratic polynomial map $F_\mu : \mathbf{R}^{m+1} \to \mathbf{R}^{u(\mu)+1}$, $u(\mu) + 1 = \{\mu_{st} > 0 \,|\, 0 \le s < t \le m\}$, with signature μ, defined by

$$F_\mu(x) = (\sqrt{\mu_{st}} x_s x_t)_{0 \le s < t \le m},$$

where we delete the zero components on the right-hand side. Universality means that, for any full quadratic polynomial map $F : \mathbf{R}^{m+1} \to \mathbf{R}^{u+1}$ with signature μ, we have $F = A \circ F_\mu$, where A is a (uniquely determined) $(u+1) \times (u(\mu)+1)$-matrix. As usual, we set

$$\langle F \rangle = A^\top A - I \in S^2(\mathbf{R}^{u(\mu)+1}),$$

and

$$\mathcal{L}_\mu = \{C \in \mathcal{E}_\mu \,|\, C + I \ge 0\},$$

where

$$\mathcal{E}_\mu = \operatorname{span}\{F_\mu(x) \odot F_\mu(x) \,|\, x \in \mathbf{R}^{m+1}\}^\perp \subset S^2(\mathbf{R}^{u(\mu)+1}).$$

A DoCarmo-Wallach type argument (Section 2.3) applies to this situation and we find that the correspondence $F \to \langle F \rangle$ parametrizes the congruence classes of full quadratic polynomial maps $F : \mathbf{R}^{m+1} \to \mathbf{R}^{u+1}$ with signature μ by the compact convex body $\mathcal{L}_\mu$ of $\mathcal{E}_\mu$. Note again that $\mathcal{L}_\mu$ is semi-algebraic. Moreover, for $\mu \in \Sigma_m^0$, the separated forms corresponding to the points of $\mathcal{L}_\mu$ are automatically eigenmaps.

Taking union with respect to all signatures, we obtain that

$$\mathcal{S}_m = \bigcup_{\mu \in \Sigma_m} \mathcal{L}_\mu \quad \left(\text{resp. } \mathcal{S}_m^0 = \bigcup_{\mu \in \Sigma_m^0} \mathcal{L}_\mu \right) \tag{2.9.9}$$

parametrizes the congruence classes of full separated forms $f : S^m \to S^n$ (resp. full separated eigenmaps $f : S^m \to S^n$). We think of $\mathcal{S}_m$ (resp. $\mathcal{S}_m^0$) as the "total space" of the "fibre bundle" with projection

$$\pi : \mathcal{S}_m \to \Sigma_m, \quad (\text{resp. } \pi^0 : \mathcal{S}_m^0 \to \Sigma_m^0)$$

associating to the congruence class of a quadratic polynomial map F its signature μ.

We now describe the "fibre" $\mathcal{L}_\mu$ over μ. Assume first that $\mu_{st} \neq 0$, for all distinct s and t. In coordinates, the condition $C \in \mathcal{E}_\mu$ gives

$$\sum_{r,s,k,l} \sqrt{\mu_{rs}\mu_{kl}} c_{rs,kl} x_r x_s x_k x_l = 0.$$

Analyzing this, we obtain that $c_{rs,kl} = 0$ if r, s, k, l are not all distinct and otherwise

$$\sqrt{\mu_{rs}\mu_{kl}} c_{rs,kl} + \sqrt{\mu_{rk}\mu_{sl}} c_{rk,sl} + \sqrt{\mu_{rl}\mu_{sk}} c_{rl,sk} = 0, \ r, s, k, l \text{ distinct.} \tag{2.9.10}$$

In particular:

$$\dim \mathcal{L}_\mu = 2\binom{m+1}{4}.$$

Assume now that $\mu_{rs} = 0$ for some $r \neq s$. Then, by the very definition of F_μ, $c_{rs,kl}$ (and, by symmetry, $c_{kl,rs}$) are nonexistent for all k, l and they are simply missing from the relations in (2.9.10). In this case $\dim \mathcal{L}_\mu \leq 2\binom{m+1}{4}$.

There is a simple geometric interpretation of (2.9.10). Assume, for simplicity, that $\mu_{rs} \neq 0$ for all $r \neq s$. Let $P_{rskl} \subset S^2(\mathbf{R}^{m(m+1)/2})$ be the three-dimensional linear subspace given by setting all entries of C other than $c_{rs,kl}$, $c_{rk,sl}$ and $c_{rl,sk}$ equal to zero. In P_{rskl}, the set of points $(c_{rs,kl}, c_{rk,sl}, c_{rl,sk})$ for which $C + I \geq 0$ holds is the cube $[-1, 1]^3$. By (2.9.10), $P_{rskl} \cap \mathcal{L}_\mu$ is the intersection of this cube with the plane through the origin with normal vector $(\sqrt{\mu_{rs}\mu_{kl}}, \sqrt{\mu_{rk}\mu_{sl}}, \sqrt{\mu_{rl}\mu_{sk}})$. The intersection is a hexagon or a quadrangle.

Note that, for $m = 3$, we have exactly four distinct indices so that the intersection $P_{rskl} \cap \mathcal{L}_\mu$ will give the entire $\mathcal{L}_\mu$.

As for dimensions, by (2.9.9), we have

$$\dim \mathcal{S}_m = 2\binom{m+1}{4} + \binom{m+1}{2}$$

and

$$\dim \mathcal{S}_m^0 = 2\binom{m+1}{4} + \binom{m+1}{2} - \binom{m+1}{1}.$$

Letting $SO(m+1)$ act on the separated forms (resp. eigenmaps), we obtain that an upper bound for the dimension of the space of congruence classes of full separable forms is $2\binom{m+1}{4}+2\binom{m+1}{2}$ (resp. for eigenmaps $2\binom{m+1}{4}+m^2-1$). In particular, for $m = 3$, the latter gives 10 as an upper bound, which coincides with the dimension of the entire moduli space $\mathcal{L}_3^2$. In fact, in the next subsection we will show that all quadratic eigenmaps $f : S^3 \to S^n$ are separable. It also follows from the dimension formula for $\mathcal{L}_m^2$ (Theorem 2.5.1) that, for $m \geq 4$, there are nonseparable quadratic eigenmaps. (For an explicit example, see Problem 2.28.)

C. Separable Eigenmaps of the 3-sphere

To illustrate the general ideas, we will work out the signature space Σ_3^0 and describe the fibre bundle $\mathcal{S}_3^0$ over Σ_3^0. Along the way we will see that some specific points of the total space correspond to classical separated eigenmaps $f : S^3 \to S^n$ for various n.

To begin with, we realize Σ_3^0 in Σ_2 by the imbedding τ. First we determine Σ_2. An element μ of Σ_2 is a symmetric (3×3)-matrix with zero diagonal entries:

$$\begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix}.$$

We identify this with the vector $(a, b, c) \in \mathbf{R}^3$. Evaluating the diagonal minors of the Gram matrix

$$G(\mu) = G(a,b,c) = \begin{bmatrix} 1 & 1-a/2 & 1-b/2 \\ 1-a/2 & 1 & 1-c/2 \\ 1-b/2 & 1-c/2 & 1 \end{bmatrix}$$

we see that $G(a, b, c) \geq 0$ iff $0 \leq a, b, c \leq 4$ and

$$a^2 + b^2 + c^2 - 2(ab + bc + ca) + abc \leq 0.$$

Thus, we derive the signature space

$$\Sigma_2 = \{(a,b,c) \in [0,4]^3 \mid a^2 + b^2 + c^2 - 2(ab + bc + ca) + abc \leq 0\}.$$

It is not hard to visualize Σ_2. Let $\mathcal{T}$ be the tetrahedron with vertices $(4,4,0)$, $(4,0,4)$, $(0,4,4)$ and the origin. Then $\mathcal{T}$ is contained in Σ_2; in

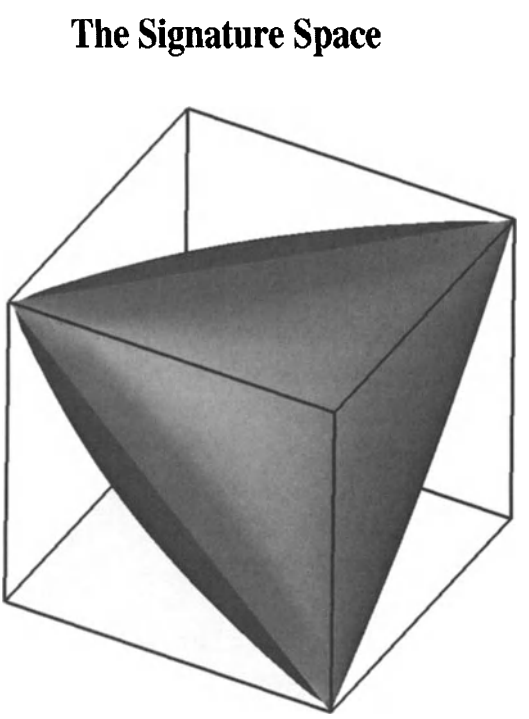


Figure 25. The signature space.

fact, the edges of $\mathcal{T}$ are the intersections of Σ_2 with the faces of the cube $[0,4]^3$. Note also that the one point slice $\{(3,3,3)\}$ is Σ_2^0 and it corresponds to the standard minimal immersion $Ver_2 : S^2 \to S^4$. Its 8/9 multiple is the orthogonal projection of $(3,3,3)$ to the center of the face of $\mathcal{T}$ opposite the origin. Σ_2 itself looks like the tetrahedron $\mathcal{T}$ "inflated" (see Figure 25).

We now determine how Σ_3^0 is imbedded in Σ_2 by τ. First of all, a typical element of Σ_3^0 is a symmetric (4×4)-matrix with zero diagonal entries which therefore has the form:

$$\begin{bmatrix} 0 & a & b & c \\ a & 0 & c & b \\ b & c & 0 & a \\ c & b & a & 0 \end{bmatrix} \tag{2.9.11}$$

with $a+b+c = 8$. Since τ deletes the last row and last column we see that it associates to this matrix the vector $(a,b,c) \in \Sigma_2$. Thus, identifying Σ_3^0 with its image, we see that Σ_3^0 is the triangle Δ with vertices $(4,4,0)$, $(4,0,4)$

and $(0,4,4)$. The center $(8/3,8/3,8/3)$ of Δ corresponds to the standard minimal immersion $Ver_3 : S^3 \to S^8$.

Let $\mu = (a,b,c) \in \Sigma_3^0$ and determine the fibre $\mathcal{L}_\mu$. Using coordinates $(x,y,u,v) \in \mathbf{R}^4$, we have

$$F_\mu(x,y,u,v) = (\sqrt{a}xy, \sqrt{b}xu, \sqrt{c}xv, \sqrt{c}yu, \sqrt{b}yv, \sqrt{a}uv). \tag{2.9.12}$$

Assume first that μ has no zero components, i.e., the corresponding point in Δ is not a vertex. Evaluating

$$\langle C, F_\mu(x) \odot F_\mu(x) \rangle = 0, \tag{2.9.13}$$

for $C \in S^2(\mathbf{R}^6)$, we find that the (6×6)-matrix C has antidiagonal entries α, β, γ, γ, β, α with $a\alpha + b\beta + c\gamma = 0$ and all other entries are zero:

$$C = \begin{bmatrix} & & & & & \alpha \\ & & & & \beta & \\ & & & \gamma & & \\ & & \gamma & & & \\ & \beta & & & & \\ \alpha & & & & & \end{bmatrix}$$

$C + I \geq 0$ translates into $(\alpha,\beta,\gamma) \in [-1,1]^3 \subset \mathbf{R}^3$. Thus $\mathcal{L}_\mu$ can be visualized as the intersection of the cube $[-1,1]^3$ with the plane through the origin with normal vector $(a,b,c) \in \Delta$. If (a,b,c) is in the interior of the triangle Δ then the intersection is a hexagon. If (a,b,c) is on an edge of Δ but it is not a vertex then the intersection is a quadrangle.

Finally, assume that (a,b,c) is one of the vertices of Δ, say $(a,b,c) = (0,4,4)$. Then the first and last components of F_μ are missing and C collapses to a (4×4)-matrix with antidiagonal entries β, γ, γ, β. Since $(\beta,\gamma) \in [-1,1]^2$, with $\beta + \gamma = 0$, we obtain that $\mathcal{L}_\mu$ is a segment parametrized by, say, $\beta \in [-1,1]$.

We can also determine the corresponding quadratic eigenmaps, or what is the same, the quadratic maps $F : \mathbf{R}^4 \to \mathbf{R}^{u+1}$ via $F = A \circ F_\mu$, where $A = \sqrt{C+I}$. (Note that F obtained this way is usually not full.) In all cases, $C + I$ is built from diagonal minors

$$\begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix}$$

with $|x| \leq 1$, whose square root is

$$\begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix}^{1/2} = \begin{bmatrix} \sqrt{1+x} + \sqrt{1-x} & \sqrt{1+x} - \sqrt{1-x} \\ \sqrt{1+x} - \sqrt{1-x} & \sqrt{1+x} + \sqrt{1-x} \end{bmatrix}.$$

Taking an appropriate representative in the equivalence class of F, we arrive at

$$F(x,y,u,v)$$

$$= \Big(\sqrt{a(1+\alpha)/2}(xy+uv), \sqrt{a(1-\alpha)/2}(xy-uv),$$
$$\sqrt{b(1+\beta)/2}(xu+yv), \sqrt{b(1-\beta)/2}(xu-yv), \qquad (2.9.14)$$
$$\sqrt{c(1+\gamma)/2}(xv+yu), \sqrt{c(1-\gamma)/2}(xv-yu) \Big).$$

The corresponding full separated eigenmaps $f : S^3 \to S^n$ can be written down explicitly. (Recall that the Gram matrix $G(\mu)$ determines the pure square part of f up to congruence.)

Here are some worked out examples.

Example 2.9.2. Let $(a, b, c) = (0, 4, 4)$ a vertex of Δ. Then $(\alpha, \beta, \gamma) = (0, 1, -1)$ gives a boundary point of the segment $\mathcal{L}_\mu$ and, up to congruence, it corresponds to the Hopf map $Hopf : S^3 \to S^2$ introduced in Section 1.4. Indeed, F in (1.9.14) is rewritten as

$$F(x, y, u, v) = (2(xu + yv), 2(xv - yu)).$$

Taking the negative of the second component and working out the associated pure part gives us the Hopf map. Note that in complex coordinates $z = x + iy$ and $w = u + iv$, we have

$$F(z, w) = 2z\bar{w}.$$

In the examples below, we will use complex terminology whenever convenient.

Example 2.9.3. Let $(a, b, c) = (0, 4, 4)$ as in the previous example and let $(\alpha, \beta, \gamma) = (0, 0, 0)$ be the center of the segment $\mathcal{L}_\mu$. We have

$$F(x, y, u, v) = (\sqrt{2}(xu + yv), \sqrt{2}(xu - yv), \sqrt{2}(xv + yu), \sqrt{2}(xv - yu)).$$

Now we replace the first two components F^0 and F^1 by $(F^0 + F^1)/\sqrt{2}$ and $(F^0 - F^1)/\sqrt{2}$ (corresponding to a reflection to the line with angle of inclination $\pi/8$ to the first axis in the coordinate plane spanned by these components). We do the same with the last two components F^2 and F^3, i.e., replace them by $(F^2 + F^3)/\sqrt{2}$ and $(F^2 - F^3)/\sqrt{2}$. This brings us to the congruent

$$F(x, y, u, v) = (2xu, 2yv, 2xv, 2yu).$$

In terms of the vectors (x, y) and (u, v) and again up to congruence, this is twice the real tensor product

$$\otimes : \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}^4$$

given by

$$\otimes((x, y), (u, v)) = (xu, xv, yu, yv).$$

The corresponding quadratic eigenmap $f_\otimes : S^3 \to S^4$ is given by

$$f_\otimes(x, y, u, v) = (x^2 + y^2 - u^2 - v^2, 2xu, 2xv, 2yu, 2yv).$$

Example 2.9.4. Let $(a,b,c) = (4,2,2)$ be the midpoint of the edge of Δ connecting the vertices $(4,0,4)$ and $(4,4,0)$. As noted above, $\mathcal{L}_\mu$ is a quadrangle. The two pairs of opposite vertices $\pm(1,-1,-1)$, resp. $\pm(0,1,-1)$ correspond to quadratic eigenmaps with $n=4$, resp. $n=5$. Let $(\alpha,\beta,\gamma) = (0,-1,1)$. Again up to congruence, the corresponding quadratic eigenmap is the complex Veronese map $Ver^{\mathbf{C}} : S^3 \to S^5$ given by

$$Ver^{\mathbf{C}}(x,y,u,v) = (x^2-y^2, u^2-v^2, \\ 2xy, \sqrt{2}(xu-yv), \sqrt{2}(xv+yu), 2uv).$$

In complex coordinates $z = x+iy$ and $w = u+iv$

$$Ver^{\mathbf{C}}(z,w) = (z^2, \sqrt{2}zw, w^2).$$

Now we will prove the main result of this subsection.

Theorem 2.9.5. *All full quadratic eigenmaps $f : S^3 \to S_V$ are separable.*

PROOF. First note that in the moduli space the separated eigenmaps fill a linear slice. (This is because the respective matrices consist of two diagonal blocks corresponding to the linear spans of $\{a_r\}_{r=0}^m$ and $\{a_{st}\}_{0\le s<t\le m}$.) In particular, "being separated" is a convex property. Let $f : S^3 \to S^n$ be a full quadratic eigenmap and set $C = \langle f \rangle \in \mathcal{L}_3^2$. Since the standard minimal immersion is separated and corresponds to the origin in $\mathcal{L}_3^2$, we may assume that $C \neq 0$. Consider the radial segment through C and let C_0 denote its intersection with $\partial\mathcal{L}_3^2$. We now recall that the opposite isometry $\gamma : (z,w) \mapsto (z,\bar{w})$ interchanges $(\mathcal{L}_3^2)^{SU(2)}$ and $(\mathcal{L}_3^2)^{SU(2)'}$ (Section 2.7). For an eigenmap $f : S^3 \to S_V$, we will denote $f' = f \circ \gamma$. We also extend this notation to the corresponding points in the moduli space: $\langle f \rangle' = \langle f' \rangle$. Finally, for $G \subset SO(4)$, we set $G' = \gamma G \gamma$. Using the notations in Section 2.7, by Theorem 2.7.5, up to an isometry on the domain, there exists a segment S with endpoints in $\mathcal{D}$ and $\mathcal{D}'$ such that $C_0 \in S$. We now use the circle groups Γ and Γ' to rotate $\mathcal{D}$ and $\mathcal{D}'$ such that the endpoints of S get to the radial segments connecting $Ver^{\mathbf{C}}$ and $Hopf_0$ in $\mathcal{D}$ and $Ver'^{\mathbf{C}}$ and $Hopf_0'$ in $\mathcal{D}'$. These are separated so that the endpoints of the rotated S are also separated. Thus, by convexity, the rotated C_0 is also separated. The rotated C is on the radial segment between 0 and C_0 so it is also separated. Hence f is separable and the proof is complete.

Remark. Using the notations above, we see that $\mathcal{S}_3^0$ can be identified with the four-dimensional slice cut out from $\mathcal{L}_3^2$ by the linear subspace spanned by the four (linearly independent) vectors $\langle Hopf_0 \rangle$, $\langle Hopf_0' \rangle$, $\langle Ver^{\mathbf{C}} \rangle$ and $\langle Ver'^{\mathbf{C}} \rangle$. Using the geometric description of $\mathcal{L}_3^2$, it is easy to see how the fibre structure $\pi^0 : \mathcal{S}_3^0 \to \Sigma_3^0$ fits in the slice. For example, the antipodal of $\langle Ver^{\mathbf{C}} \rangle$ in $\partial\mathcal{L}_3^2$ is $\langle Hopf \rangle$. Similarly, the antipodal of $\langle Ver'^{\mathbf{C}} \rangle$ is $\langle Hopf' \rangle$ and the three segments connecting $\langle Hopf_0 \rangle$, $\langle Hopf_{\pi/2} \rangle$ and $\langle Hopf \rangle$ with their conjugates correspond to the three fibres $\mathcal{L}_\mu$ at the three vertices μ of the triangle Δ. The interior of the segment with endpoints $\langle Hopf_0 \rangle$ and

$\langle Hopf'_{\pi/2} \rangle$ corresponds to the trace of a vertex of the quadrangle $\mathcal{L}_\mu$ as μ slides along an edge of Δ.

Problems

2.1. Given $\xi \in \mathcal{P}^p_{m+1}$, write

$$\xi = \xi_0 + x_m \xi_1 + \frac{x_m^2}{2!}\xi_2 + \ldots + \frac{x_m^p}{p!}\xi_p,$$

where ξ_r, $r = 0, \ldots, p$, do not depend on x_m.
(a) Show that $\xi_0 = \xi|_{\mathbf{R}^m}$ and $\xi_1 = \partial_m \xi|_{\mathbf{R}^m}$, where $\mathbf{R}^m = x_m^\perp$.
(b) Assuming $\xi \in \mathcal{H}^p_m$, show that, if ξ_0 and ξ_1 are given, there is a unique ξ satisfying the decomposition above.
(c) Conclude that the correspondence $\xi \mapsto (\xi_0, \xi_1)$ defines an isomorphism

$$\mathcal{H}^p_m|_{SO(m)} \cong \mathcal{P}^p_m \oplus \mathcal{P}^{p-1}_m$$

of $SO(m)$-modules.
(d) Use (2.1.14) to derive (2.1.15).

2.2. Use Lemma 2.1.4 to show that the scalar product (1.4.3) of $\xi_1, \xi_2 \in W_p \subset \mathcal{H}^p_3$ can be written as

$$\langle \xi_1, \xi_2 \rangle_{W_p} = \frac{(p+1)!}{\mathrm{vol}\,(S^3)} \int_{S^3} \Re(\xi_1 \bar{\xi}_2) v_{S^3}, \quad \mathrm{vol}\,(S^3) = 2\pi^2.$$

Compare this with (2.1.27) for $m = 3$. (Hint: Perform the differentiation in $\triangle^p \Re(\xi_1 \bar{\xi}_2)$, integrate over S^3, apply Lemma 2.1.4, and finally use (1.4.4).)

2.3. Show that the p-eigenmap $f_p : S^3 \to S_{W_p}$ defined by

$$f_p = \frac{1}{\sqrt{p+1}} \left(\frac{1}{\sqrt{(p-q)!q!}} f_{z^{p-q}w^q} \right)_{q=0}^{p}$$

is congruent to the standard minimal immersion $f_{3,p}$. Here $f_{z^{p-q}w^q}$ is the orbit map obtained by the equivariant construction applied to $z^{p-q}w^q \in W_p$ (Section 1.4). (Hint: Use Schur's orthogonality relations applied to the orthonormal basis

$$\left\{ \frac{1}{\sqrt{(p-q)!q!}} z^{p-q} w^q \right\}_{q=0}^{p} \subset W_p$$

to conclude that the components of f_p are orthogonal with the same norm.)

2.4. Derive the formula

$$[\partial_a, \delta_b] = [\partial_b, \delta_a] + \frac{4p}{\lambda_{2p}} E_{ab}$$

on $\mathcal{H}^p$.

2.5. Let $a, b \in \mathbf{R}^{m+1}$ be orthogonal unit vectors. Let $R_{\theta,ab} \in SO(m+1)$ denote the positive rotation with angle θ in the plane spanned by the vectors a and b

(and identity in the orthogonal complement). Show that

$$E_{ab}\xi = \frac{d}{d\theta}\xi \circ R^{-1}_{\theta,ab}|_{\theta=0}.$$

2.6. Let $p \geq q$. Define $F : \mathcal{H}^p \otimes \mathcal{H}^q \to \mathcal{P}^{p+q}$ by $F(\chi_1 \otimes \chi_2) = \chi_1 \cdot \chi_2$, $\chi_1 \in \mathcal{H}^p$, $\chi_2 \in \mathcal{H}^q$.
(a) Show that F is a homomorphism of $SO(m+1)$-modules.
(b) Use (2.1.7) and Schur's lemma to show that the image of F is $\mathcal{H}^{p+q} \oplus \mathcal{H}^{p+q-2}\rho^2 \oplus \ldots \oplus \mathcal{H}^{p-q}\rho^{2q}$.
(c) Conclude that $\mathcal{H}^{p+q} \oplus \mathcal{H}^{p+q-2} \oplus \ldots \oplus \mathcal{H}^{p-q}$ is an $SO(m+1)$-submodule of $\mathcal{H}^p \otimes \mathcal{H}^q$. (For a complete decomposition of $\mathcal{H}^p \otimes \mathcal{H}^q$ into irreducible components, see (3.1.5)-(3.1.7).)

2.7. Consider the generalized Veronese map $\mathrm{Ver}_d : S^2 \to S^{2d}$ defined in (1.4.11).
(a) Indicate the dependence of Ver_d on $x \in S^2 \subset \mathbf{R}^3$ and on $(z,w) \in \mathbf{C}^2$ by writing $\mathrm{Ver}_d(x)(z,w)$. Use (2.1.26) (for $m=3$) to derive the formula

$$\partial_{\mathrm{Ver}_d(x)(z,w)}\mathrm{Ver}_d(x)(z',w') = \frac{1}{2^d d!}(zw' - z'w)^{2d}.$$

(b) Use (2.1.26) again to show that, with respect to the orthonormal basis (1.4.6), the components of $\mathrm{Ver}_d(x)$ are orthogonal with the same norm.

2.8. Prove that a full p-eigenmap $f : S^m \to S_V$ has orthonormal components iff the image of the standard minimal immersion $f_p : S^m \to S_{\mathcal{H}^p}$ is contained in the product of spheres

$$\sqrt{\frac{\dim V}{\dim \mathcal{H}^p}}S_{V_f} \times \sqrt{\frac{\dim V^\perp}{\dim \mathcal{H}^p}}S_{V_f^\perp} \subset S_{\mathcal{H}^p}.$$

2.9. Let $f : S^m \to S_V$ and $f' : S^m \to S_{V'}$ be full p-eigenmaps with $f' \leftarrow f$. Let $A : V \to V'$ be a surjective linear map defined by $f' = A \circ f$. Show that the linear map of V_f to $V_{f'}$ that corresponds to A under the isomorphisms $V \cong V_f$ and $V' \cong V_{f'}$ is $\langle f' \rangle_f + I_V$.

2.10. **(a)** Let $f : \mathbf{R}^{m+1} \to V$ be a harmonic p-homogeneous polynomial map. Show that $\langle f \rangle \in \mathcal{K}^p$ is traceless iff

$$\int_{S^m} \sum_{j=0}^{n} (f^j)^2 v_{S^m} = \mathrm{vol}\,(S^m),$$

where f^j, $j = 0, \ldots, n$, are the components of f relative to an orthonormal basis in V.
(b) Apply the DoCarmo-Wallach argument to the linear slice

$$\mathcal{K}_0^p = \mathcal{K}^p \cap S_0^2(\mathcal{H}^p),$$

where $S_0^2(\mathcal{H}^p)$ is the traceless part of $S^2(\mathcal{H}^p)$. Conclude that $\mathcal{K}_0^p$ is compact.

2.11. Let $f : S^m \to S_V$ be a full p-eigenmap. Write $f_1 : S^m \to S^n$ and $f_2 : S^m \to S^n$ the full p-eigenmaps obtained from f by fixing two orthonormal bases in V. Show that for the Gram matrices we have $G(f_2) = UG(f_1)U^\top$, where $U \in O(n+1)$ is the transfer matrix between the two bases.

2.12. Use (2.3.12) to show that the DoCarmo-Wallach parametrization is injective on the congruence classes of full p-eigenmaps. (Hint: If $\langle f_1\rangle = \langle f_2\rangle \in \mathcal{L}^p$ then $V_{f_1} = V_{f_2}$ and $G(f_1)$ and $G(f_2)$ are conjugate. To show that f_1 and f_2 are congruent, use the fact that, in general, the Gram matrix determines the basis up to an isometry. See also Problem 2.11.)

2.13. Extend the orthonormal basis (2.3.11) to an orthonormal basis in $\mathcal{H}^p$, and write the standard minimal immersion f_p in terms of this extended basis. Show that

$$\sqrt{C+I}f_p(x) = f(x), \quad x \in S^m.$$

(Compare this with the proof of Theorem 2.3.1.)

2.14. Show that a full p-eigenmap $f : S^m \to S_V$ has orthonormal components iff $\langle f\rangle + I$ is $\lambda_0 + 1$ times the orthogonal projection to V_f iff

$$\langle f\rangle^2 = (\lambda_0 - 1)\langle f\rangle + \lambda_0 I,$$

where $\lambda_0 = \dim \mathcal{H}^p / \dim V - 1$ is the largest eigenvalue of $\langle f\rangle$.

2.15. Show by explicit computation that $\mathcal{L}_{Hopf}$ is trivial, and conclude that the Hopf map is linearly rigid.

2.16. Show by explicit computation that

$$\mathcal{E}_{Ver\mathbf{C}} = \{C(a,b) \in S^2(\mathbf{R}^6) \,|\, a,b \in \mathbf{R}\},$$

where

$$C(a,b) = \begin{bmatrix} 0 & 0 & 0 & 0 & a & -b \\ 0 & 0 & 0 & 0 & -b & -a \\ 0 & 0 & -a & b & 0 & 0 \\ 0 & 0 & b & a & 0 & 0 \\ a & -b & 0 & 0 & 0 & 0 \\ -b & -a & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Show that $C(a,b) + I_6 \geq 0$ iff $a^2 + b^2 \leq 1$, and conclude that $\mathcal{L}_{Ver\mathbf{C}}$ is a flat 2-disk.

2.17. Give an example of an eigenmap $f : S^m \to S_V$ for which $f^{\pm} : S^m \to S_{V\otimes\mathcal{H}^1}$ are not full.

2.18. Give an example of a p-eigenmap $f : S^m \to S_V$ such that $\langle f\rangle \in \partial\mathcal{L}^p$, but $\langle f^{\pm}\rangle \in \operatorname{int}\mathcal{L}^{p\pm1}$.

2.19. **(a)** Extend the degree-raising and -lowering operators to harmonic p-homogeneous polynomial maps $f : \mathbf{R}^{m+1} \to V$.
(b) For a linear subspace $V \subset \mathcal{H}^p$, define

$$\partial V = \{\partial_a \chi \,|\, \chi \in V, a \in \mathbf{R}^{m+1}\}$$

and

$$\delta V = \{\delta_a \chi \,|\, \chi \in V, a \in \mathbf{R}^{m+1}\}.$$

Show that

$$V_{f+} = \delta V_f \quad \text{and} \quad V_{f-} = \partial V_f,$$

for any harmonic p-homogeneous polynomial map $f : \mathbf{R}^{m+1} \to V$.
(c) Differentiate (2.1.8), and derive the relation

$$\partial\delta V = \delta\partial V,$$

for any linear subspace $V \subset \mathcal{H}^p$. (Hint: First derive the inclusions $V \subset \partial\delta V$ and $V \subset \delta\partial V$.)
(d) Use (c) to prove that

$$(\Phi_3^- \circ \Phi_2^+)(\mathcal{L}_3^2) \subset \operatorname{int} \mathcal{L}_3^2.$$

2.20. Prove (2.3.7) using the following steps.
(a) Given a full p-eigenmap $f : S^m \to S_V$, show that

$$\Psi^0(\langle v, f\rangle \odot \langle w, f\rangle)(x) = \langle v, f(x)\rangle\langle w, f(x)\rangle.$$

(For the definition of Ψ^0, see Section 2.5.)
(b) Use $\mathcal{E}^p = \ker \Psi^0$ to show that, under the isomorphism $S^2V \cong S^2(V_f)$ induced by $V \cong V_f$, $C \in \mathcal{E}_f \subset S^2V$ corresponds to an element C_f in $\mathcal{E}^p(\cap S^2(V_f))$.

2.21. Prove that a full nonstandard quadratic eigenmap $f : S^3 \to S_V$ has orthonormal components iff, up to an isometry on the domain, f is congruent to the Hopf map or the complex Veronese map. Conclude that the congruence classes of full quadratic eigenmaps $f : S^3 \to S_V$ with orthonormal components fill an antipodal pair of $SU(2)'$-orbits in $(\mathcal{L}_3^2)^{SU(2)}$ scaled by distortion 2, and another pair in $(\mathcal{L}_3^2)^{SU(2)'}$ and the origin in $\mathcal{L}_3^2 \subset \mathcal{E}_3^2$.

2.22. Use Lemma 2.8.1 to derive the orthogonality relation (2.1.12) for ultraspherical polynomials for $a \in \mathbf{Z}/2$.

2.23. Give an alternative proof of Corollary 2.7.2 by analyzing the possible configurations of the feasible system of vectors in $\mathbf{R}^3$ associated to quadratic eigenmaps $f : S^3 \to S^2$.

2.24. As in Examples 2.9.2-2.9.4, let $(a, b, c) = (4, 2, 2)$. Show that the interior of the quadrangle $\mathcal{L}_\mu$ corresponds to quadratic eigenmaps with spherical range dimension $n = 7$. In particular, work out the quadratic eigenmap that corresponds to the center $(\alpha, \beta, \gamma) = (0, 0, 0)$.

2.25. As in Examples 2.9.2-2.9.4, let $(a, b, c) = (2, 3, 3)$. Show that the interior of the hexagon $\mathcal{L}_\mu$ corresponds to quadratic eigenmaps with spherical range dimension $n = 8$, the edges to $n = 7$, and the vertices to $n = 6$. Work out the quadratic eigenmap that corresponds to $(\alpha, \beta, \gamma) = (0, 1, -1)$.

2.26. Let $f : S^m \to S_V$ be a full separated eigenmap with associated feasible set of vectors $\{a_r\}_{r=0}^m \cup \{a_{st}\}_{0 \le s < t \le m}$. We define the *rank* of f as

$$\operatorname{rank} f = \dim \operatorname{span} \{a_r\}_{r=0}^m.$$

Let $f : S^m \to S_V$ have rank 1.
(a) Show that m is odd and, up to congruence, the feasible set of vectors satisfy

$$a_0 = \ldots = a_{m_0-1} = -a_{m_0} = \ldots = -a_{2m_0-1}, \quad m_0 = \frac{m+1}{2},$$

and the signature μ is

$$\mu_{rs} = 2(1 - \langle a_r, a_s \rangle) \begin{cases} 4 \text{ if } 0 < |r-s| \leq l \\ 0 \text{ if } |r-s| > l. \end{cases}$$

(b) Let $F : \mathbf{R}^{2m_0} \to W$ denote the quadratic polynomial map associated to f. Use (a) to conclude that $F : \mathbf{R}^{m_0} \times \mathbf{R}^{m_0} \to W$, $\mathbf{R}^{2m_0} = \mathbf{R}^{m_0} \times \mathbf{R}^{m_0}$, is bilinear and satisfies

$$|(1/2)F(x,y)|^2 = |x|^2|y|^2, \quad x, y \in \mathbf{R}^{m_0}.$$

(c) Let W be a Euclidean vector space. An *orthogonal multiplication of type* (a, b, W) is a bilinear map $G : \mathbf{R}^a \times \mathbf{R}^b \to W$ which is *normed*:

$$|G(x,y)|^2 = |x|^2|y|^2, \quad x \in \mathbf{R}^a, \ y \in \mathbf{R}^b.$$

Let G be an orthogonal multiplication of type (m_0, m_0, W). The *Hopf-Whitehead construction* associates to G the separated eigenmap

$$f_G : S^{2m_0-1} \to S_{\mathbf{R} \oplus W}$$

defined by

$$f_G(x,y) = (|x|^2 - |y|^2, 2G(x,y)), \quad x, y \in \mathbf{R}^{m_0}.$$

Use (a)-(b) to show that, up to congruence, a rank 1 separated eigenmap is obtained from an orthogonal multiplication by the Hopf-Whitehead construction.

2.27. Consider the quaternionic multiplication as an orthogonal multiplication R of type $(4, 4, \mathbf{H})$ (Problem 2.26). The Hopf-Whitehead construction associates to G the quaternionic Hopf map $\mathrm{Hopf}^{\mathbf{H}}$: $S^7 \to S^4$. Show the existence of full quartic (4)-eigenmaps in the form $Hopf^{\mathbf{H}} \circ \tilde{f} : S^7 \to S^{n+5}$, $n = 2, 3 - 8$, where $f : S^3 \to S^n$ is a full quadratic eigenmap (and the domain dimension has been raised). (In particular, for $n = 2$, we can obtain a full quartic eigenmap of S^7 into itself.)

2.28. (a) Let $f : S^m \to S^n$ be a separated eigenmap. Use the definition of the rank in Problem 2.26 to show that

$$\operatorname{rank} f \left(1 + \frac{m+1}{\operatorname{rank} f + 1} \right) \leq n + 1.$$

(Hint: Write f in the form (2.9.1). Consider the relation $\sim$ on $\{0, \ldots, m\}$ defined by $r \sim s$ if $a_{rs} = 0$ and show that it is an equivalence. Let $C_1, \ldots, C_t$ denote the equivalence classes and set $z_l = |C_l|$, $l = 1, \ldots, t$. By the definition of $\sim$, there are exactly t distinct vectors in $\{a_0, \ldots, a_m\}$. Verify that they are linearly dependent, and rank $f + 1 \leq t$. On the other hand, $\sum_{l=1}^{t} z_l = m + 1$ so that

$$tz_1 = t \min \{z_l \mid l = 1, \ldots, t\} \leq m + 1,$$

where we assumed that the minimum is attained at z_1. Let $r_l \in C_l$ and consider the system of vectors $\{a_{r_l s}\}_{s=0}^{m} \subset \mathbf{R}^{u+1}$. Show that this system is orthogonal so that its nonzero vectors, of which there are $m + 1 - z_l$ in number, form a linearly independent system. Finally, verify that $m + 1 - z_l \leq v + 1$, $u + v = n$, and use $tz_1 \leq m + 1$ to obtain $(m+1)(1 - 1/t) \leq v + 1$.)

(b) A p-homogeneous polynomial $\xi : \mathbf{R}^{m+1} \to \mathbf{R}$ is called an *eiconal* if

$$| \bigtriangledown \xi |^2 = \rho^{2(p-1)}.$$

In particular, the restriction $f = \nabla\xi|_{S^m}$ is a $(p-1)$-form $f : S^m \to S^m$. Verify that the following gives cubic eiconals ξ: Let x, y be real and X, Y, Z real, complex, quaternion or octonian. Then $\xi : \mathbf{R}^{2+3\nu} \to \mathbf{R}$, $\nu = 1, 2, 4, 8$, is given by

$$\xi = \frac{1}{3}x^3 - xy^2 + \frac{1}{2}(|X|^2 + |Y|^2 - 2|Z|^2) + \frac{\sqrt{3}}{2}y(|X|^2 - |Y|^2) + \frac{\sqrt{3}}{2}(XYZ + \bar{Z}\bar{Y}\bar{X}).$$

Use (a) to show that the gradient of a cubic eiconal on S^4 is a nonseparable quadratic eigenmap $f : S^4 \to S^4$.

3

Moduli for Spherical Minimal Immersions

3.1 Conformal Eigenmaps and Moduli

Let V be a Euclidean vector space. A map $f : S^m \to S_V$ is *conformal* if

$$\langle f_*(X_x), f_*(Y_x)\rangle = c(x)\langle X_x, Y_x\rangle, \quad X_x, Y_x \in T_x(S^m), \; x \in S^m, \qquad (3.1.1)$$

for some positive function $c : S^m \to \mathbf{R}$, called the *conformality factor.*

Proposition 3.1.1. *Let f be a conformal p-eigenmap. Then the conformality factor c is λ_p/m, and f is an isometric minimal immersion of the m-sphere S^m_κ of constant curvature $\kappa = m/\lambda_p$ into the unit sphere S_V (of curvature 1). Conversely, if $f : S^m_\kappa \to S_V$ is an isometric minimal immersion then $\kappa = m/\lambda_p$ for some p and, keeping the original (curvature 1) metric on the domain, $f : S^m \to S_V$ is a conformal p-eigenmap with conformality factor $c = \lambda_p/m$.*

PROOF. By Proposition A.2.1 from Appendix 2, $f : S^m \to S_V$ is a p-eigenmap iff it is a harmonic map with energy density λ_p. Since the energy density is the norm square of the differential f_*, assuming that f is a conformal p-eigenmap, the conformality factor is λ_p/m. Changing the domain S^m to S^m_κ, where $\kappa = m/\lambda_p$, f becomes an isometric immersion. We now use the fact that an isometric immersion is minimal iff it is harmonic (cf. Appendix 2). The converse follows by reversing the steps above.

We wish to parametrize the space of congruence classes of full isometric minimal immersions $f : S^m_\kappa \to S_V$. It will be more convenient to keep the original metric on the domain. We will consider *homothetic* minimal

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immersions $f : S^m \to S_V$, i.e. instead of isometry, we require

$$\langle f_*(X), f_*(Y)\rangle = \frac{\lambda_p}{m}\langle X, Y\rangle, \tag{3.1.2}$$

for all vector fields X and Y on S^m. The factor λ_p/m is called the *homothety constant of* f. We call a homothetic minimal immersion $f : S^m \to S_V$ with homothety λ_p/m a *(spherical) minimal immersion of (algebraic) degree* p. We suppress the degree whenever there is no danger of confusion.

By Proposition 3.1.1 above, our task is equivalent to classifying the congruence classes of full minimal immersions $f : S^m \to S_V$ of degree p.

Let $f : S^m \to S_V$ be a full minimal immersion of degree p. As in the case of eigenmaps, we can give a parametrization of the congruence classes of full minimal immersions $f' : S^m \to S_{V'}$ of degree p that are derived from f. To do this, we define

$$\mathcal{F}_f = \{\, f_*(X)^\vee \odot f_*(Y)^\vee \,|\, X, Y \in T(S^m)\}^\perp \subset S^2V.$$

Equivalently, we set

$$\mathcal{F}_f = \{C \in S^2V \,|\, \langle Cf_*(X)^\vee, f_*(Y)^\vee\rangle = 0,\ X, Y \in T(S^m)\}.$$

Here and in what follows, it is understood that X and Y belong to the same tangent space of S^m, and $^\vee : TV \to V$ denotes the canonical identification of vectors tangent to V with vectors in V.

Proposition 3.1.2. *Let $f : S^m \to S_V$ be a full minimal immersion of degree p. Then we have*

$$\mathcal{F}_f \subset \mathcal{E}_f.$$

PROOF. Let $C \in \mathcal{F}_f$ and choose $\varepsilon > 0$ such that $\varepsilon C + I > 0$. Define $g = \sqrt{\varepsilon C + I} \circ f : S^m \to V$. Since the components of f are in $\mathcal{H}^p$, g is a harmonic p-homogeneous polynomial map. We claim that g is homothetic with homothety λ_p/m. Indeed, for $X, Y \in T(S^m)$, we have

$$\begin{aligned}\langle g_*(X), g_*(Y)\rangle &= \langle(\varepsilon C + I)f_*(X)^\vee, f_*(Y)^\vee\rangle \\ &= \varepsilon\langle Cf_*(X)^\vee, f_*(Y)^\vee\rangle + \langle f_*(X), f_*(Y)\rangle \\ &= \frac{\lambda_p}{m}\langle X, Y\rangle.\end{aligned}$$

Applying Proposition A.2.3 from Appendix 2, we conclude that the image of g is contained in S_V. Hence

$$1 = |g|^2 = |\sqrt{\varepsilon C + I} \circ f|^2 = \langle(\varepsilon C + I) \circ f, f\rangle = \varepsilon\langle C \cdot f, f\rangle + 1.$$

$\langle Cf, f\rangle = 0$ follows. This means that $C \in \mathcal{E}_f$ (Section 2.3). The proposition follows.

Finally, let

$$\mathcal{M}_f = \{C \in \mathcal{F}_f \,|\, C + I \geq 0\}.$$

The defining relation $C + I \geq 0$ for $\mathcal{M}_f$ in $\mathcal{F}_f$ is the same as for $\mathcal{L}_f$ in $\mathcal{E}_f$. Thus, Proposition 3.1.2 gives us

$$\mathcal{M}_f = \mathcal{F}_f \cap \mathcal{L}_f.$$

Being a linear slice of $\mathcal{L}_f$, $\mathcal{M}_f$ is a convex body in $\mathcal{F}_f$. By Corollary 2.3.3, $\mathcal{M}_f$ is compact. The analogue of Theorem 2.3.1 is the following:

Theorem 3.1.3. *Given a full minimal immersion $f : S^m \to S_V$ of degree p, the set of congruence classes of full minimal immersions $f' : S^m \to S_{V'}$ of degree p that are derived from f can be parametrized by the convex body $\mathcal{M}_f$. The parametrization is given by associating to the congruence class of f' the endomorphism $\langle f' \rangle_f = A^\top A - I \in S^2V$, where $f' = A \circ f$ and $A : V \to V'$ is linear and surjective.*

PROOF. A p-eigenmap is a minimal immersion of degree p iff it is homothetic. Hence we need to show that $\mathcal{M}_f \subset \mathcal{L}_f$ parametrizes those full p-eigenmaps $f' : S^m \to S_{V'}$ derived from f that are homothetic. Let $f' = A \circ f$ with $A : V \to V'$ being linear and surjective. For $C = A^\top A - I \in S^2V$, we have

$$\begin{aligned}\langle C, f_*(X)^\check{} \odot f_*(Y)^\check{} \rangle &= \langle C f_*(X)^\check{}, f_*(Y)^\check{} \rangle \\ &= \langle A \cdot f_*(X)^\check{}, A \cdot f_*(Y)^\check{} \rangle - \langle f_*(X), f_*(Y) \rangle \\ &= \langle f'_*(X), f'_*(Y) \rangle - \frac{\lambda_p}{m} \langle X, Y \rangle.\end{aligned}$$

This is zero for all $X, Y \in T(S^m)$, iff $C \in \mathcal{F}_f$ iff $f' : S^m \to S_{V'}$ is homothetic.

The convex body $\mathcal{M}_f$ is said to be the relative *moduli space* associated to the full minimal immersion $f : S^m \to S_V$.

Recall the standard minimal immersion $f_p : S^m \to S_{\mathcal{H}^p}$, defined by (2.2.1). Since the components of f_p form an orthonormal basis in $\mathcal{H}^p$, the *standard moduli space* $\mathcal{M}^p = \mathcal{M}^p_m = \mathcal{M}_{f_p}$ parametrizes the congruence classes of *all* full minimal immersions of degree p. As usual, we set $\langle f \rangle = \langle f \rangle_{f_p}$. The interior of $\mathcal{M}^p$ parametrizes the full minimal immersions of degree p with maximal range ($\cong \mathcal{H}^p$). We call a full minimal immersion $f : S^m \to S_V$ of degree p of *boundary type* if $\langle f \rangle \in \partial \mathcal{M}^p$, or equivalently, if $\dim V < \dim \mathcal{H}^p$.

Let $f : S^m \to S_V$ and $f' : S^m \to S_{V'}$ be full minimal immersions of degree p. If f' is derived from f then the imbedding $\iota : \mathcal{L}_{f'} \to \mathcal{L}_f$ defined in (2.3.4) restricts to an imbedding $\iota : \mathcal{M}_{f'} \to \mathcal{M}_f$.

As in the case of eigenmaps, given full minimal immersions $f : S^m \to S_V$ and $f' : S^m \to S_{V'}$ of degree p with $f' \leftarrow f$, $f' = A \circ f$, $A : V \to V'$, for the linearized map $\iota_0 : S^2V' \to S^2V$ defined in Section 2.3, we have $C' \in \mathcal{F}_{f'}$ iff $\iota_0(C') \in \mathcal{F}_f$. Indeed

$$\begin{aligned}\langle \iota_0(C') f_*(X)^\check{}, f_*(Y)^\check{} \rangle &= \langle (A^\top C' A) f_*(X)^\check{}, f_*(Y)^\check{} \rangle \\ &= \langle C' f'_*(X)^\check{}, f'_*(Y)^\check{} \rangle,\end{aligned}$$

and the claim follows. Thus, we have the linear isomorphism

$$\mathcal{F}_{f'} \cong S^2(V_{f'}) \cap \mathcal{F}_f, \tag{3.1.3}$$

and the absolute version

$$\mathcal{F}_f \cong S^2(V_f) \cap \mathcal{F}^p. \tag{3.1.4}$$

A full minimal immersion is said to be *linearly rigid* if its moduli space $\mathcal{M}_f$ is trivial. (3.1.4) implies that a full minimal immersion $f : S^m \to S_V$ of degree p is linearly rigid iff $S^2(V_f)$ and $\mathcal{F}^p$ have trivial intersection in $S^2(\mathcal{H}^p)$.

Replacing p-eigenmaps with minimal immersions of degree p, Theorems 2.3.5, 2.3.7, 2.3.8, and 2.3.19 are valid for minimal immersions with some obvious modifications. For later reference we include here the analogue of Theorem 2.3.7.

Theorem 3.1.4. *Given a full minimal immersion $f : S^m \to S_V$ of degree p, the moduli space $\mathcal{M}_f$ is the convex hull of the points that correspond to linearly rigid full minimal immersions derived from f. Moreover, the set of extremal points is minimal in the sense that the convex hull of any proper subset of extremal points is a proper subset of $\mathcal{M}_f$.*

Remark 1. This is a followup to the remark after Corollary 2.3.9. Recall that a symmetric positive semidefinite bilinear form $G \in S^2(\mathcal{H}^p)$ is the eigenform of a p-eigenmap iff

$$G(\delta_x, \delta_x) = 1, \quad x \in S^m.$$

The composition

$$S^m \xrightarrow{\delta} (\mathcal{H}^p)^* \xrightarrow{\pi} (\mathcal{H}^p)^*/\ker G,$$

is an eigenmap whose eigenform is G where $(\mathcal{H}^p)^*/\ker G$ is endowed with the positive definite scalar product induced by G. Homothety (3.1.2), imposed on this composition, is the condition

$$G(X\delta, Y\delta) = \frac{\lambda_p}{m}\langle X, Y\rangle,$$

for all vector fields X, Y on S^m. Here, for $x \in S^m$, $X_x\delta \in (\mathcal{H}^p)^*$ is the linear functional $\chi \mapsto X_x\chi$. From this we obtain that the moduli $\mathcal{M}^p$ can be parametrized by positive semidefinite bilinear forms $G \in S^2(\mathcal{H}^p)$ satisfying

$$\begin{aligned} G(\delta_x, \delta_x) &= 1, \quad x \in S^m, \\ G(X_x\delta, Y_x\delta) &= \frac{\lambda_p}{m}\langle X_x, Y_x\rangle, \quad X_x, Y_x \in T_x(S^m),\ x \in S^m. \end{aligned}$$

Geometrically, $\mathcal{M}^p$ can be viewed as the intersection of the *positive cone* $P^+\mathcal{H}^p = \{G \in S^2(\mathcal{H}^p) \mid G \geq 0\}$ with an affine subspace in $S^2(\mathcal{H}^p)$ modeled

on the linear subspace of solutions $G_0 \in S^2(\mathcal{H}^p)$ of the equations

$$\begin{aligned} G_0(\delta_x, \delta_x) &= 0, \quad x \in S^m, \\ G_0(X_x\delta, Y_x\delta) &= 0, \quad X_x, Y_x \in T_x(S^m),\ x \in S^m. \end{aligned}$$

Remark 2. Weingart [1] applies Theorem 2.3.5 to conclude that there exist spherical minimal immersions *whose images are not imbedded submanifolds.* A brief outline of his argument follows:

Let $f_1 : S^m \to S_{V_1}$ and $f_2 : S^m \to S_{V_2}$ be full minimal immersions of degree p, and assume that the invariance groups G_1 and G_2 of f_1 and f_2 are finite subgroups of $SO(m+1)$ both acting without fixed points on S^m. We claim that the minimal immersion $f : S^m \to S_{V_1 \times V_2}$ of degree p in Theorem 2.3.5 ($\langle f \rangle = \lambda_1\langle f_1 \rangle + \lambda_2\langle f_2 \rangle$, $\lambda_1 + \lambda_2 = 1$, $0 < \lambda_1, \lambda_2 < 1$) has a smooth imbedded image iff any nontrivial element in the finite set $G_1G_2 \subset SO(m+1)$ acts on S^m without fixed points. Using the list of spherical minimal immersions in Section 1.5, it is easy to give examples of subgroups $G_1 \subset SU(2)$ and $G_2 \subset SU(2)'$ in $SO(4) = SU(2) \cdot SU(2)'$ (G_1 and G_2 can actually be chosen conjugate) such that G_1G_2 has fixed points on S^3. Applying the equivariant construction to absolute invariants of G_1 and G_2 will then give f_1 and f_2, and thus a spherical minimal immersion f whose image is *not* an imbedded submanifold.

To prove the claim, recall that, up to congruence, f_1 and f_2 can be recovered from their eigenforms G_{f_1} and G_{f_2} via the compositions

$$S^m \xrightarrow{\delta} (\mathcal{H}^p)^* \xrightarrow{\pi_1} (\mathcal{H}^p)^*/\ker G_{f_1},$$

and

$$S^m \xrightarrow{\delta} (\mathcal{H}^p)^* \xrightarrow{\pi_2} (\mathcal{H}^p)^*/\ker G_{f_2}.$$

The minimal immersion $f : S^m \to S_{V_1 \times V_2}$ of degree p above has eigenform $G_f = \lambda_1 G_{f_1} + \lambda_2 G_{f_2}$. Since $0 < \lambda_1, \lambda_2 < 1$, we have $\ker G_f = \ker G_{f_1} \cap \ker G_{f_2}$. Thus, f is congruent to the composition

$$f : S^m \xrightarrow{\delta} (\mathcal{H}^p)^* \xrightarrow{\pi} (\mathcal{H}^p)^*/\ker G_{f_1} \cap \ker G_{f_2}.$$

Now consider the short exact sequence

$$\begin{aligned} 0 \to (\mathcal{H}^p)^*/(\ker G_{f_1} \cap \ker G_{f_2}) &\to (\mathcal{H}^p)^*/\ker G_{f_1} \oplus (\mathcal{H}^p)^*/\ker G_{f_2} \\ &\to (\mathcal{H}^p)^*/(\ker G_{f_1} + \ker G_{f_2}) \to 0, \end{aligned}$$

where $\alpha + (\ker G_{f_1} \cap \ker G_{f_2})$ is mapped to $(\alpha + \ker G_{f_1}) \oplus (\alpha + \ker G_{f_2})$, and $(\alpha_1 + \ker G_{f_1}) \oplus (\alpha_2 + \ker G_{f_2})$ is mapped to $\alpha_1 - \alpha_2 + (\ker G_{f_1} + \ker G_{f_2})$. Comparing this sequence with the representations of f_1, f_2, and f above, we see that $f(S^m)$ is a smooth imbedded submanifold in $(\mathcal{H}^p)^*/(\ker G_{f_1} \cap \ker G_{f_2})$ iff $(f_1, f_2)(S^m)$ is a smooth imbedded submanifold in $(\mathcal{H}^p)^*/\ker G_{f_1} \oplus (\mathcal{H}^p)^*/\ker G_{f_2}$, iff the image of the diagonal

immersion

$$\Delta : S^m \to S^m/G_1 \times S^m/G_2, \quad \Delta(x) = (G_1(x), G_2(x)), \ x \in S^m$$

is a smooth imbedded submanifold.

To finish the proof, we claim that the image of Δ is a smooth imbedded submanifold iff G_1G_2 has no nontrivial isometry with fixed point in S^m. First, notice that $\Delta(p) = \Delta(p')$ iff $p' = g_1(p) = g_2(p)$ for some $g_1 \in G_1$ and $g_2 \in G_2$ iff $g_1^{-1}g_2 \in G_1G_2$ fixes p. If G_1G_2 has no nontrivial isometry with fixed point in S^m then, with the notation above, $\Delta(p) = \Delta(p')$ implies that $g_1^{-1}g_2$ is trivial, i.e. $g_1 = g_2 \in G_1 \cap G_2$. Thus, $\Delta(S^m)$ is diffeomorphic with $S^m/(G_1 \cap G_2)$, a smooth manifold. Finally, assume that $g_1^{-1}g_2 \in G_1G_2$ is a nontrivial isometry that fixes $p \in S^m$, and let $p' = g_1(p) = g_2(p)$. We claim that $\Delta(p) = \Delta(p')$ is not a smooth point of $\Delta(S^m)$. Assume, on the contrary, that it is. Then, for the tangent spaces T_p and $T_{p'}$ (with S^m suppressed), we have $\Delta_*(T_p) = \Delta_*(T_{p'})$. We can factor Δ as $(\pi_1 \times \pi_2) \circ \Delta_0$, where $\Delta_0 : S^m \to S^m \times S^m$ is the diagonal map, and $\pi_1 : S^m \to S^m/G_1$ and $\pi_2 : S^m \to S^m/G_2$ are the orbit maps. Since G_1 and G_2 are finite, π_1 and π_2 are finite coverings, and $(\pi_1 \times \pi_2)_*$, evaluated at (p,p) and at (p',p') identifies $T_p \times T_p$ and $T_{p'} \times T_{p'}$. Since $g_1 \in G_1$ and $g_2 \in G_2$, we have $\pi_1 \circ g_1 = \pi_1$ and $\pi_2 \circ g_2 = \pi_2$. By assumption, $\Delta_*(T_p) = \Delta_*(T_{p'})$ so that, under $(g_1 \times g_2)_*$, the diagonals $(\Delta_0)_*(T_p) \subset T_p \times T_p$ and $(\Delta_0)_*(T_{p'}) \subset T_{p'} \times T_{p'}$ must correspond to each other. This is, however, a contradiction since

$$\{((g_1^{-1})_*(X'), (g_2^{-1})_*(X')) \,|\, X' \in T_{p'}\} = \{((g_1^{-1}g_2)_*X, X) \,|\, X \in T_p\}$$

cannot be equal to $\{(X,X) \,|\, X \in T_p\}$ as $g_1^{-1}g_2$ is not the identity. The claim follows.

On the moduli space $\mathcal{L}^p$, the $SO(m+1)$-action

$$g \cdot \langle f \rangle = \langle f \circ g^{-1} \rangle, \quad g \in SO(m+1), \ \langle f \rangle \in \mathcal{L}^p,$$

leaves $\mathcal{M}^p$ invariant. (When precomposing eigenmaps with isometries, the property of being homothetic remains invariant.) On the other hand, the linear span of $\mathcal{M}^p$ is $\mathcal{F}^p$. It follows that $\mathcal{F}^p$ is an $SO(m+1)$-submodule of $\mathcal{E}^p \subset S^2(\mathcal{H}^p)$ (Proposition 3.1.2), where the $SO(m+1)$-module structure on $S^2(\mathcal{H}^p)$ is defined in Section 2.3.

Our goal is to determine the $SO(m+1)$-module structure of $\mathcal{F}^p$, or what amounts to the same thing, to decompose $\mathcal{F}^p$ into irreducible submodules. Once the irreducible components of $\mathcal{F}^p$ are known, as a byproduct, we can compute $\dim \mathcal{M}^p = \dim \mathcal{F}^p$ by the Weyl dimension formula (Appendix 3). This will lead to the main result of this chapter: the exact determination of the dimension of the standard moduli space $\mathcal{M}^p$.

In order to state the main result we now recall some finer results of the representation theory of $\mathcal{H}^p \otimes \mathcal{H}^q$, $p \geq q$. (For more details, see Appendix 3.)

First of all, we have the recurrence formula

$$\mathcal{H}^p \otimes \mathcal{H}^q = \sum_{r=0}^{q} V^{(p+q-r,r,0,\dots,0)} \oplus (\mathcal{H}^{p-1} \otimes \mathcal{H}^{q-1}),\ p \geq q \geq 1,\ m \geq 3. \tag{3.1.5}$$

Here $V^{(u_1,\dots,u_d)} = V_{m+1}^{(u_1,\dots,u_d)}$, $d = [(m+1)/2]$, denotes the (unique) complex irreducible $SO(m+1)$-module with highest weight vector $(u_1, \dots, u_d)$ relative to the standard maximal torus in $SO(m+1)$. Recall that, for $m = 3$ and $v > 0$, $V_4^{(u,v)}$ needs to be replaced by $V_4^{(u,v)} \oplus V_4^{(u,-v)}$.

For example

$$\mathcal{H}^p = V^{(p,0,\dots,0)}, \tag{3.1.6}$$

as $SO(m+1)$-modules. For notational simplicity, unless stated otherwise, we denote a real (absolutely irreducible) representation and its (irreducible) complexification by the same symbol.

Iterating (3.1.5), we get

$$\mathcal{H}^p \otimes \mathcal{H}^q = \sum_{(u,v)\in\triangle_0^{p,q}\,;\,u+v\equiv p+q\,(\mathrm{mod}\,2)} V^{(u,v,0,\dots,0)},\ p \geq q \geq 1,\ m \geq 3, \tag{3.1.7}$$

where $\triangle_0^{p,q}$ is the closed convex triangle in $\mathbf{R}^2$ with vertices $(p-q,0)$, (p,q) and $(p+q,0)$.

Setting $p = q$, we have

$$\mathcal{H}^p \otimes \mathcal{H}^p = S^2(\mathcal{H}^p) \oplus \wedge^2(\mathcal{H}^p) = \sum_{(u,v)\in\triangle_0^p\,;\,u+v\ \mathrm{even}} V^{(u,v,0,\dots,0)},\ p \geq 1,\ m \geq 3,$$

where we simplified the notation by setting $\triangle_0^p = \triangle_0^{p,p}$, the closed convex triangle in $\mathbf{R}^2$ with vertices $(0,0)$, (p,p) and $(2p,0)$. Then $V^{(u,v,0,\dots,0)}$ belongs to $S^2(\mathcal{H}^p)$ (resp. $\wedge^2(\mathcal{H}^p)$) iff u and v are both even (resp. both odd). We obtain

$$S^2(\mathcal{H}^p) = \sum_{(u,v)\in\triangle_0^p;\,u,v\ \mathrm{even}} V^{(u,v,0,\dots,0)}. \tag{3.1.8}$$

With this we can now reformulate Theorem 2.5.1, as follows: According to (2.1.14), as $SO(m+1)$-modules, we have

$$\mathcal{P}^{2p} = \sum_{l=0}^{p} \mathcal{H}^{2l} = \sum_{l=0}^{p} V^{(2l,0,\dots,0)},$$

where in the second equality we used (3.1.6). The irreducible modules in this sum correspond to the even coordinate points of the base of the triangle $\triangle_0^p$. By Theorem 2.5.1, $\mathcal{E}^p = S^2(\mathcal{H}^p)/\mathcal{P}^{2p}$. Factoring out with these components amounts to slicing off the base of $\triangle_0^p$. Thus

$$\mathcal{E}^p = \sum_{(u,v)\in\triangle_1^p;\,u,v\ \mathrm{even}} V^{(u,v,0,\dots,0)}, \tag{3.1.9}$$

where $\triangle_1^p$ is the closed convex triangle in $\mathbf{R}^2$ with vertices $(2,2)$, (p,p) and $(2p-2,2)$.

By Theorem 2.6.1, $\Phi_p^+ : \mathcal{E}^p \to \mathcal{E}^{p+1}$ is an $SO(m+1)$-equivariant imbedding. In view of (3.1.9), this corresponds to the inclusion $\triangle_1^p \subset \triangle_1^{p+1}$. Also by Theorem 2.6.1, $\Phi_p^- : \mathcal{E}^p \to \mathcal{E}^{p-1}$ is onto. Since $\triangle_1^p$ is $\triangle_1^{p-1}$ plus the northeast side of $\triangle_1^p$, we obtain the following:

Corollary 3.1.5. *We have*

$$\ker \Phi_p^- = \sum_{j=0}^{[p/2]} V^{(2p-2j,2j,0,\ldots,0)}.$$

Remark. Let $f : S^m \to S_V$ be a full p-eigenmap and assume that $\langle f \rangle \in \sum_{j=0}^{[p/2]} V^{(2p-2j,2j,0,\ldots,0)}$. By Corollary 3.1.5, $\langle f^- \rangle = \Phi^-(\langle f \rangle) = 0$, so that $f^- : S^m \to S_{V \otimes \mathcal{H}^1}$ (made full) is the standard minimal immersion $f_{p-1} : S^m \to S_{\mathcal{H}^{p-1}}$ of degree $p-1$. In particular, we have

$$\dim V \otimes \mathcal{H}^1 \geq \dim \mathcal{H}^{p-1}.$$

Equivalently,

$$\dim V \geq \frac{\dim \mathcal{H}^{p-1}}{\dim \mathcal{H}^1} = \frac{(2p+m-3)(p+m-3)!}{(p-1)!(m-1)!(m+1)},$$

where we used (2.1.5). For fixed m, the lower bound is

$$\frac{(2p+m-3)(p+m-3)(p+m-4)\cdots p}{(m-1)!(m+1)} = O(p^{m-1}) \text{ as } p \to \infty.$$

On the other hand, as noted at the end of Section 2.5, if f is linearly rigid then

$$\dim V \leq O(p^{m/2}) \text{ as } p \to \infty.$$

Combining these, we obtain that, for $m \geq 3$ and p large, a full p-eigenmap $f : S^m \to S_V$ with $\langle f \rangle \in \sum_{j=0}^{[p/2]} V^{(2p-2j,2j,0,\ldots,0)}$ cannot be linearly rigid. Geometrically, this means that the slice

$$\mathcal{L}^p \cap \sum_{j=0}^{[p/2]} V^{(2p-2j,2j,0,\ldots,0)}$$

does not contain any extremal points of $\mathcal{L}^p$ (Lemma 2.3.6).

The main purpose of this chapter is to give an affirmative answer to the *exact dimension conjecture*, i.e. to prove the following structure theorem for $\mathcal{F}^p$:

Theorem 3.1.6. *For $m \geq 3$ and $p \geq 4$,*

$$V^{(2,2,0,\ldots,0)}, \ldots, V^{(2p-2,2,0,\ldots,0)} \tag{3.1.10}$$

are not components of $\mathcal{F}^p$, *so that we have*

$$\mathcal{F}^p = \sum_{(u,v)\in\triangle_2^p;u,v \text{ even}} V^{(u,v,0,\ldots,0)}, \tag{3.1.11}$$

where $\triangle_2^p$ *is the closed convex triangle in* $\mathbf{R}^2$ *with vertices* $(4,4)$, (p,p) *and* $(2p-4,4)$.

Although technically more involved, the proof of Theorem 3.1.6 follows the pattern of the proof of Theorem 2.5.1. Indeed, to locate $\mathcal{F}^p$ within $\mathcal{E}^p$, we will construct an $SO(m+1)$-module homomorphism Ψ from $\mathcal{E}^p$ with kernel $\mathcal{F}^p$. Roughly speaking, for a p-eigenmap $f : S^m \to S^n$, $\Psi(\langle f\rangle)$ will measure how far f is from being homothetic. To show that Ψ is nonzero on the components (3.1.10), we will use rigidity and induction with respect to the degree. In the main induction step, the degree-raising and -lowering operators will play crucial roles.

Remark. The main result of the theory of DoCarmo-Wallach [1] is to show that $\mathcal{F}^p$ contains the right-hand side of (3.1.11). This enabled them to conclude that the moduli $\mathcal{M}_m^p$ is nontrivial iff $m \geq 3$ and $p \geq 4$, in fact they proved that, for these ranges,

$$\dim \mathcal{M}_m^p = \dim \mathcal{F}_m^p \geq \sum_{(u,v)\in\triangle_2^p;u,v \text{ even}} \dim_{\mathbf{C}} V^{(u,v,0,\ldots,0)} (\geq 18).$$

They believed that this inequality was actually an equality. (See DoCarmo-Wallach [1], 1.6. Remark, p.44, and 5.10. Remark, p.56.) This is precisely the content of Theorem 3.1.6. To obtain their lower bound, DoCarmo and Wallach used the decomposition of the normal bundle of f_p into osculating bundles, interpreted minimality in representation theoretical terms, and finally made use of induced representations along with Frobenius reciprocity. Our "operator approach" follows a different path. In fact, the only overlap between the DoCarmo-Wallach treatment and the one given here is the use of the decompositions (3.1.7)-(3.1.8). Note also that in the first nonrigid range $m = 3$ and $p = 4$, the equality

$$\dim \mathcal{M}_3^4 = \dim \mathcal{F}_3^4 = 18$$

has been obtained by Muto [1] by enormous but explicit tensor computations. As noted in the previous chapter, a proof of the simpler decomposition formula (3.1.9) is implicitly contained in DoCarmo-Wallach [1]. (They did not define the concept of eigenmaps introduced essentially in Eells-Sampson [1].) Note finally, that Weingart [1] recently gave a new proof of Theorem 3.1.6. His proof is more algebraic in character than the one we will present here.

3.2 Conformal Fields and Eigenmaps

Let $f : S^m \to S_V$ be a full p-eigenmap. We define $\Psi(f) = \Psi_p(f)$, a symmetric 2-tensor on S^m, by

$$\Psi(f)(X,Y) = \langle f_*(X), f_*(Y)\rangle - \frac{\lambda_p}{m}\langle X, Y\rangle, \tag{3.2.1}$$

where X and Y are vector fields on S^m. By (3.1.2), $\Psi(f)$ measures how far f is from being homothetic (thereby minimal). In particular, $\Psi(f) = 0$ iff f is homothetic. In particular, $\Psi(f_p) = 0$, since the standard minimal immersion $f_p : S^m \to S_{\mathcal{H}^p}$ is homothetic. It will also be convenient to write (3.2.1) in "relative form:"

$$\Psi(f)(X,Y) = \langle f_*(X), f_*(Y)\rangle - \langle (f_p)_*(X), (f_p)_*(Y)\rangle,$$

especially when we compare f with f_p. The definition of $\Psi(f)$ makes sense for any p-homogeneous polynomial map $f : \mathbf{R}^{m+1} \to V$, and $\Psi(f)$ is extended as a symmetric 2-tensor on $\mathbf{R}^{m+1}$ by

$$\begin{aligned}\Psi(f)(X,Y) &= \langle f_*(X), f_*(Y)\rangle - \frac{\lambda_p}{m}\langle X, Y\rangle \rho^{2(p-1)} \\ &= \langle f_*(X), f_*(Y)\rangle - \langle (f_p)_*(X), (f_p)_*(Y)\rangle, \\ &\quad \rho^2 = |x|^2 = x_0^2 + \ldots x_m^2,\ x = (x_0, \ldots, x_m) \in \mathbf{R}^{m+1}.\end{aligned}$$

We now restrict $\Psi(f)$ to the finite dimensional vector space of conformal fields on S^m. For $a \in \mathbf{R}^{m+1}$, the *conformal field* X^a, a vector field on S^m, is defined by

$$(X^a_x)^{\vee} = a - a^* \cdot x = a - \langle a, x\rangle x, \quad x \in S^m.$$

Geometrically, X^a is the constant vector field a on $\mathbf{R}^{m+1}$ along the inclusion $S^m \subset \mathbf{R}^{m+1}$ followed by projection to the tangent bundle $T(S^m)$. It is clear that the conformal fields span (pointwise) each tangent space on S^m. Setting

$$\Psi(f)(a,b) = \Psi(f)(X^a, X^b), \quad a, b \in \mathbf{R}^{m+1},$$

we obtain that $\Psi(f)(a,b) = 0$ for all $a, b \in \mathbf{R}^{m+1}$ iff f is homothetic.

Just as p-eigenmaps can be considered as harmonic p-forms, a conformal field X^a can also be considered as a vector field on $\mathbf{R}^{m+1}$ with the extension

$$(X^a_x)^{\vee} = a - \frac{\langle a, x\rangle}{\rho^2}x, \quad x \in \mathbf{R}^{m+1}. \tag{3.2.2}$$

With these extensions, we have the following:

Lemma 3.2.1. *For $a, b \in \mathbf{R}^{m+1}$, $\Psi(f)(a,b)$ is a $2(p-1)$-homogeneous polynomial, that is, $\Psi(f)(a,b) \in \mathcal{P}^{2(p-1)}$.*

PROOF OF LEMMA 3.2.1. The crucial computational formula for $\Psi(f)$ which will be used repeatedly is:

$$\Psi(f)(a,b) = -\langle f, \partial_a \partial_b f\rangle \tag{3.2.3}$$

$$+p(p-1)\left(\left(1+\frac{1}{m}\right)a^*b^*-\frac{1}{m}\langle a,b\rangle\rho^2\right)\rho^{2(p-2)}.$$

Notice that our lemma follows from this since the right-hand side belongs to $\mathcal{P}^{2(p-1)}$.

For the proof of (3.2.3), we first note that

$$f_*(X^a)^\vee = \partial_a f - p\frac{a^*}{\rho^2}f,$$

where $^\vee : TV \to V$ denotes the canonical identification given by translating tangent vectors to the origin. Indeed, using (3.2.2) and homogeneity, we compute

$$f_*(X^a)^\vee = \partial_{(X^a)^\vee} f = \partial_a f - \frac{\langle a,x\rangle}{\rho^2}\partial_x f = \partial_a f - p\frac{a^*}{\rho^2}f.$$

We also have

$$\langle f, \partial_a f\rangle = \frac{1}{2}\partial_a |f|^2 = \frac{1}{2}\partial_a \rho^{2p} = pa^*\rho^{2(p-1)}.$$

Using these we compute

$$\begin{aligned}
\Psi(f)(a,b) &= \langle f_*(X^a)^\vee, f_*(X^b)^\vee\rangle - \frac{\lambda_p}{m}\langle (X^a)^\vee, (X^b)^\vee\rangle\rho^{2(p-1)} \\
&= \left\langle \partial_a f - p\frac{a^*}{\rho^2}f, \partial_b f - p\frac{b^*}{\rho^2}f\right\rangle \\
&\quad - \frac{\lambda_p}{m}\left\langle a - \frac{a^*}{\rho^2}x, b - \frac{b^*}{\rho^2}x\right\rangle\rho^{2(p-1)} \\
&= \langle \partial_a f, \partial_b f\rangle \qquad (3.2.4)\\
&\quad + \left(\frac{\lambda_p}{m} - p^2\right)a^*b^*\rho^{2(p-2)} - \frac{\lambda_p}{m}\langle a,b\rangle\rho^{2(p-1)}.
\end{aligned}$$

Now

$$\langle \partial_a f, \partial_b f\rangle = -\langle f, \partial_a\partial_b f\rangle + \partial_a\langle f, \partial_b f\rangle, \qquad (3.2.5)$$

where the second term on the right-hand side rewrites as

$$\begin{aligned}
\partial_a\langle f, \partial_b f\rangle &= p\partial_a\left(b^*\rho^{2(p-1)}\right) \\
&= p\langle a,b\rangle\rho^{2(p-1)} + 2p(p-1)a^*b^*\rho^{2(p-2)}.
\end{aligned}$$

Putting all these together, we arrive at (3.2.3).

Remark. The coefficient of $\rho^{2(p-2)}$ on the right-hand side of (3.2.3) is *harmonic* so that the canonical decomposition of the polynomial $\Psi(f)(a,b)$ can be obtained from the canonical decomposition of $\langle f, \partial_a\partial_b f\rangle$.

Lemma 3.2.1 says that, for any full p-eigenmap $f : S^m \to S_V$, $\Psi(f)$ defines a symmetric bilinear map

$$\Psi(f) : \mathbf{R}^{m+1} \times \mathbf{R}^{m+1} \to \mathcal{P}^{2(p-1)}.$$

It is clear from the definition that $\Psi(f)$ depends only on the congruence class of f. More explicitly, letting $f = A \circ f_p$, $A : V \to V'$ being linear and surjective, and using the relative form of $\Psi(f)$, we compute

$$\begin{aligned}\Psi(f)(X,Y) &= \Psi(f)(X,Y) - \Psi(f_p)(X,Y)\\ &= \langle A(f_p)_*(X)^{\check{}}, A(f_p)_*(Y)^{\check{}}\rangle\\ &\quad -\langle (f_p)_*(X)^{\check{}}, (f_p)_*(Y)^{\check{}}\rangle\\ &= \langle (A^{\top}A - I)(f_p)_*(X)^{\check{}}, (f_p)_*(Y)^{\check{}}\rangle\\ &= \langle \langle f\rangle (f_p)_*(X)^{\check{}}, (f_p)_*(Y)^{\check{}}\rangle.\end{aligned}$$

Thus, for $C = \langle f\rangle \in \mathcal{L}^p$, we have

$$\Psi(f)(X,Y) = \langle C(f_p)_*(X)^{\check{}}, (f_p)_*(Y)^{\check{}}\rangle.$$

This enables us to extend Ψ to $\mathcal{E}^p \subset S^2(\mathcal{H}^p)$ as follows. For $C \in \mathcal{E}^p$, we define

$$\Psi(C) : \mathbf{R}^{m+1} \times \mathbf{R}^{m+1} \to \mathcal{P}^{2(p-1)}$$

by

$$\Psi(C)(a,b) = \langle C(f_p)_*(X^a)^{\check{}}, (f_p)_*(X^b)^{\check{}}\rangle, \quad a, b \in \mathbf{R}^{m+1}.$$

Lemma 3.2.2. *We have*

$$\begin{aligned}\Psi(C)(a,b) &= \langle \partial_a f, \partial_b f\rangle - \langle \partial_a f_p, \partial_b f_p\rangle\\ &= \langle \partial_a C f_p, \partial_b f_p\rangle. \qquad (3.2.6)\end{aligned}$$

PROOF. Since $\mathcal{L}^p$ is a convex body in $\mathcal{E}^p$, we may assume that $C = \langle f\rangle$ for a full p-eigenmap $f : S^m \to S_V$. As noted above, $\Psi(f)(a,b)$ is the same as the difference $\Psi(f)(a,b) - \Psi(f_p)(a,b)$. Now (3.2.6) follows from (3.2.3) and (3.2.5) when we notice that the last two terms on the right-hand side of (3.2.3) do not depend on f and thereby cancel:

$$\begin{aligned}\Psi(f)(a,b) &= \langle \partial_a f, \partial_b f\rangle - \langle \partial_a f_p, \partial_b f_p\rangle\\ &= \langle \partial_a A f_p, \partial_b A f_p\rangle - \langle \partial_a f_p, \partial_b f_p\rangle\\ &= \langle \partial_a C f_p, \partial_b f_p\rangle.\end{aligned}$$

Lemma 3.2.3. *Let P and R be finite dimensional vector spaces and $\mathcal{B} : R \times R \times R \to P$ a trilinear map. Assume that $\mathcal{B}$ is skew symmetric in the first two variables and symmetric in the last two variables. Then $\mathcal{B}$ vanishes identically.*

PROOF. The permutation $[(12)(23)]^3 = (132)^3$ is the identity. Letting it act on the variables of $\mathcal{B}$, by the assumed symmetries, $\mathcal{B}$ will change sign three times. The lemma follows.

Remark. Lemma 3.2.3 is known as the Braid lemma (Berger [1]). The name comes from the fact that, in the proof, the transpositions applied to the arguments of the tensor $\mathcal{B}$ correspond to crossings of the three strands in a braid.

Theorem 3.2.4. *Let $f : S^m \to S_V$ be a minimal immersion of degree p. If $p \leq 3$, then f is congruent to the standard minimal immersion.*

Remark. This result is due to DoCarmo-Wallach [1] (1.4. Theorem, p. 44). For a generalization to any analytic domain, see Wallach [1] (Proposition 11.1, p. 32.) Rather than referring to this general result we prefer to give a simple proof here using our framework and terminology, and pointing out a connection to the Braid lemma.

PROOF OF THEOREM 3.2.4. We have

$$\Psi^0(f) = \langle Cf_p, f_p \rangle = 0, \tag{3.2.7}$$

$$\Psi(f)(a,b) = \langle \partial_a Cf_p, \partial_b f_p \rangle = 0, \tag{3.2.8}$$

where the first equality is because f is spherical, and the second is because f is homothetic (Lemma 3.2.2). Differentiating (3.2.7) and using the symmetry of C, we obtain:

$$\langle Cf_p, \partial_a f_p \rangle = 0. \tag{3.2.9}$$

Consider the trilinear map

$$\mathcal{B} : \mathbf{R}^{m+1} \times \mathbf{R}^{m+1} \times \mathbf{R}^{m+1} \to \mathcal{P}^{2p-3}$$

defined by

$$\mathcal{B}(a,b,c) = \langle \partial_a Cf_p, \partial_b \partial_c f_p \rangle, \ a,b,c \in \mathbf{R}^{m+1}.$$

Since directional derivatives commute, $\mathcal{B}$ is symmetric in the last two variables. Differentiating (3.2.8), we see that $\mathcal{B}$ is skew symmetric in the first two variables. By Lemma 3.2.3, $\mathcal{B}$ vanishes identically:

$$\langle \partial_a Cf_p, \partial_b \partial_c f_p \rangle = 0. \tag{3.2.10}$$

Differentiating (3.2.9) and using (3.2.8), we have

$$\langle Cf_p, \partial_a \partial_b f_p \rangle = 0. \tag{3.2.11}$$

Differentiating this again and using (3.2.10), we finally obtain

$$\langle Cf_p, \partial_a \partial_b \partial_c f_p \rangle = 0. \tag{3.2.12}$$

Turning to the proof, we first observe that the statement is clearly true for $p = 1$ so that we may assume $p = 2$ or $p = 3$. Writing (3.2.11) and (3.2.12)

in coordinates, we get

$$\sum_{j,j'=0}^{N(p)} c_{jj'} f_p^j \partial_a \partial_b f_p^{j'} = 0$$

and

$$\sum_{j,j'=0}^{N(p)} c_{jj'} f_p^j \partial_a \partial_b \partial_c f_p^{j'} = 0.$$

Here, depending on whether $p = 2$ or $p = 3$, the second and third derivatives are constants so that linear independence of the components f_p^j give

$$\sum_{j'=0}^{N(p)} c_{jj'} \partial_a \partial_b f_p^{j'} = \partial_a \partial_b \sum_{j'=0}^{N(p)} c_{jj'} f_p^{j'} = 0, \quad j = 0, \ldots, N(p),$$

or

$$\sum_{j'=0}^{N(p)} c_{jj'} \partial_a \partial_b \partial_c f_p^{j'} = \partial_a \partial_b \partial_c \sum_{j'=0}^{N(p)} c_{jj'} f_p^{j'} = 0, \quad j = 0, \ldots, N(p).$$

By homogeneity of the components $f_p^{j'}$, we can remove the derivatives in both cases and obtain

$$\sum_{j'=0}^{N(p)} c_{jj'} f_p^{j'} = 0, \quad j = 0, \ldots, N(p).$$

Once again, linear independence of the components f_j' gives $c_{jj'} = 0$. Thus $C = 0$, and the proof is complete.

Remark. Notice that, by homogeneity, (3.2.7) follows from (3.2.8) (Problem 3.2).

Lemma 3.2.5. trace $\Psi(C) = 0$.

PROOF. With respect to the standard orthonormal basis $\{e_i\}_{i=0}^m \subset \mathbf{R}^{m+1}$, we have

$$\operatorname{trace} \Psi(C) = \sum_{i=0}^m \Psi(C)(e_i, e_i) = \sum_{i=0}^m \langle \partial_i C f_p, \partial_i f_p \rangle = \frac{1}{2} \triangle \langle C f_p, f_p \rangle = 0.$$

For $C \in \mathcal{E}^p$, $\Psi(C)$ defines a linear map

$$\Psi(C) : S^2(\mathbf{R}^{m+1}) \to \mathcal{P}^{2(p-1)},$$

where we used symmetry of $\Psi(C)$ with respect to its vectorial arguments. The trivial summand $\mathbf{R}$ in the decomposition

$$S^2(\mathbf{R}^{m+1}) = \mathbf{R} \oplus S_0^2(\mathbf{R}^{m+1})$$

corresponds to the trace (Example 2.5.4), and $S_0^2(\mathbf{R}^{m+1})$ is the traceless part of $S^2(\mathbf{R}^{m+1})$. By Lemma 3.2.5, $\Psi(C)$ is zero on $\mathbf{R}$. Restricting, we arrive at the linear map

$$\Psi(C) : S_0^2(\mathbf{R}^{m+1}) \to \mathcal{P}^{2(p-1)}$$

Since $S_0^2(\mathbf{R}^{m+1})^* \cong S_0^2(\mathcal{H}^1) \cong \mathcal{H}^2$, we will consider $\Psi(C)$ as an element of $\mathcal{P}^{2(p-1)} \otimes \mathcal{H}^2$. The correspondence $C \mapsto \Psi(C)$ thus defines a linear map

$$\Psi : \mathcal{E}^p \to \mathcal{P}^{2(p-1)} \otimes \mathcal{H}^2. \tag{3.2.13}$$

By definition, we have

$$\ker \Psi = \mathcal{F}^p.$$

The following lemma states that Ψ is a homomorphism of $SO(m+1)$-modules:

Lemma 3.2.6. *For $g \in SO(m+1)$, we have*

$$\Psi(g \cdot C)(g \cdot a, g \cdot b) = \Psi(C)(a, b) \circ g^{-1}. \tag{3.2.14}$$

PROOF. This is a direct consequence of the $SO(m+1)$-equivariance of f_p. We can take $C = \langle f \rangle$, where $f : S^m \to S_V$ is a p-eigenmap. Then (3.2.14) is rewritten as

$$\Psi(f \circ g^{-1})(g \cdot a, g \cdot b) = \Psi(f)(a, b) \circ g^{-1}. \tag{3.2.15}$$

The transformation rule for conformal fields is

$$\begin{aligned} g_*^{-1}(X^{ga})_x\check{} &= g^{-1}(ga - \langle ga, x\rangle x) \\ &= a - \langle a, g^{-1}x\rangle g^{-1}x = (X^a)\check{}_{g^{-1}x}, \quad x \in S^m. \end{aligned}$$

This is obvious since X^a is the projection of the uniform vector field a (on $\mathbf{R}^{m+1}$) to $T(S^m)$. With this, (3.2.15) follows.

Remark. In close analogy with the meaning of Ψ^0, as multiplication (cf. the remark after Example 2.5.4) there is an algebraic interpretation of Ψ, as follows: Let $\Pi_{p,q} : \mathcal{P}^p \otimes \mathcal{P}^q \to \mathcal{P}^{p+q-2} \otimes \mathcal{P}^2$ be the homomorphism of $SO(m+1)$-modules defined by

$$\Pi_{p,q}(\xi \odot \eta)(a \otimes b) = \frac{1}{2}(\partial_a \xi \cdot \partial_b \eta + \partial_b \xi \cdot \partial_a \eta), \ \xi \in \mathcal{P}^p, \eta \in \mathcal{P}^q, \ a, b \in \mathbf{R}^{m+1},$$

where $(\mathbf{R}^{m+1})^* = \mathcal{H}^1$ and $S^2(\mathcal{H}^1) = \mathcal{H}^0 \oplus \mathcal{H}^2 = \mathcal{P}^2$ (Example 2.5.4 and (2.1.14)). In coordinates, we have

$$\Pi_{p,q}(\xi \otimes \eta) = \sum_{i,k=0}^{m} (\partial_i \xi \cdot \partial_k \eta) \otimes y_i y_k.$$

We claim that

$$\Pi_{p,p}|_{S^2(\mathcal{H}^p)} = \Psi,$$

where $S^2(\mathcal{H}^p)$ is considered as a linear subspace in $\mathcal{H}^p \otimes \mathcal{H}^p \subset \mathcal{P}^p \otimes \mathcal{P}^p$. Writing everything in coordinates, we let $\{f_p^j\}_{j=0}^{N(p)} \subset \mathcal{H}^p$ be an orthonormal basis, and $\sum_{j,l=0}^{N(p)} c_{jl} f_p^j \otimes f_p^l \in S^2(\mathcal{H}^p)$ a typical element identified with the matrix $C = (c_{jl})_{j,l=0}^{N(p)}$. By (3.2.6), we have

$$\begin{aligned}
\Psi(C)(a,b)(x) &= \langle \partial_a C f_p, \partial_b f_p \rangle(x) \\
&= \left\langle \sum_{j=0}^{N(p)} (\partial_a f_p^j)(x) C f_p^j, \sum_{l=0}^{N(p)} (\partial_b f_p^l)(x) f_p^l \right\rangle \\
&= \sum_{j,l=0}^{N(p)} \langle C f_p^j, f_p^l \rangle (\partial_a f_p^j)(x)(\partial_b f_p^l)(x) \\
&= \sum_{j,l=0}^{N(p)} c_{jl} (\partial_a f_p^j)(x)(\partial_b f_p^l)(x) \\
&= \Pi_{p,p}\left(\sum_{j,l=0}^{N(p)} c_{jl} f_p^j \otimes f_p^l \right)(a,b)(x).
\end{aligned}$$

The claim follows.

Π^0 (Section 2.5) and Π are related by the identity

$$\Pi^0_{p+q-2,2} \circ \Pi_{p,q} = pq \Pi^0_{p,q},$$

an immediate consequence of the homogeneity of the polynomial arguments. Setting $p = q$ and restricting to $S^2(\mathcal{H}^p)$, we obtain

$$\Pi^0_{2(p-1),2} \circ \Psi = p^2 \Psi^0.$$

This can be expressed in the commutativity of the diagram

$$\begin{array}{ccc}
\mathcal{E}^p & \xrightarrow{\Psi} & \mathcal{P}^{2(p-1)} \otimes \mathcal{H}^2 \\
\downarrow & & \downarrow \Pi^0_{2(p-1),2} \\
S^2(\mathcal{H}^p) & \xrightarrow{p^2 \Psi^0} & \mathcal{P}^{2p}
\end{array}$$

where the left vertical arrow is the inclusion.

We now decompose the range $\mathcal{P}^{2(p-1)} \otimes \mathcal{H}^2$ of Ψ into irreducible $SO(m+1)$-modules. By the canonical decomposition (2.1.4), we can write

$$\mathcal{P}^{2(p-1)} = \sum_{l=0}^{p-1} \mathcal{H}^{2l} \cdot \rho^{2(p-l-1)}, \tag{3.2.16}$$

and hence, as $SO(m+1)$-modules, we have

$$\mathcal{P}^{2(p-1)} \otimes \mathcal{H}^2 = \sum_{l=0}^{p-1} \mathcal{H}^{2l} \otimes \mathcal{H}^2 = \mathcal{H}^2 \oplus \sum_{l=1}^{p-1} \mathcal{H}^{2l} \otimes \mathcal{H}^2.$$

By (3.1.7), each term in the last sum decomposes as

$$\mathcal{H}^{2l} \otimes \mathcal{H}^2 = \sum_{(u,v)\in\triangle_0^{(2l,2)};\, u,v \text{ even}} V^{(u,v,0,\ldots,0)},\quad l \geq 1, \tag{3.2.17}$$

where $\triangle_0^{(2l,2)}$ has vertices $(2l-2,0)$, $(2l,2)$ and $(2l+2,0)$. The only common term in the decomposition of $\mathcal{E}^p$ in (3.1.9) and (3.2.17) is $V^{(2l,2,0,\ldots,0)}$. Hence $V^{(u,v,0,\ldots,0)} \subset \mathcal{E}^p$ with $v \geq 4$ must belong to the kernel of Ψ. Since the latter is $\mathcal{F}^p$, we obtain

$$\mathcal{F}^p \supset \sum_{(u,v)\in\triangle_2^p;\, u,v \text{ even}} V^{(u,v,0,\ldots,0)}. \tag{3.2.18}$$

Remark. This lower estimate is the main result of DoCarmo-Wallach [1]. Notice that it follows almost immediately from our setup.

We also see that, corresponding to the even coordinate points of the base of the triangle $\triangle_1^p$, for $l = 1,\ldots,p-1$, $V^{(2l,2,0,\ldots,0)} \not\subset \mathcal{F}^p$ *iff* Ψ *is injective on* $V^{(2l,2,0,\ldots,0)}$, *iff*

$$\Psi_p|_{V^{(2l,2,0,\ldots,0)}} \neq 0,\quad l = 1,\ldots,p-1. \tag{3.2.19}$$

To complete the proof of Theorem 3.1.6, this is exactly what remains to be shown.

Let $C \in \mathcal{E}^p$ and decompose

$$C = \sum_{(u,v)\in\triangle_1^p;\, u,v \text{ even}} C^{u,v}$$

as in (3.1.9). By (3.2.18), we have

$$\Psi(C) = \sum_{l=1}^{p-1} \Psi(C^{2l,2}) \in \mathcal{P}^{2(p-1)}.$$

According to (3.2.16), for $a, b \in \mathbf{R}^{m+1}$, $\Psi(C^{2l,2})(a,b)$ is a harmonic homogeneous polynomial of degree $2l$ multiplied by $\rho^{2(p-l-1)}$. Summarizing, we arrive at the following:

Theorem 3.2.7. *Given $C \in \mathcal{E}^p$ we consider the canonical decomposition*

$$\Psi(C)(a,b) = \sum_{l=1}^{p-1} h^l(a,b)\rho^{2(p-l-1)},\quad h^l(a,b) \in \mathcal{H}^{2l},\ l = 1,\ldots,p-1. \tag{3.2.20}$$

We have the implication

$$h^l(a,b) \neq 0 \text{ for some } a,b \in \mathbf{R}^{m+1} \Rightarrow V^{(2l,2,0,\ldots,0)} \not\subset \mathcal{F}^p. \tag{3.2.21}$$

Given $C \in \mathcal{E}^p$, h^l is called the *l-th canonical coefficient of* C, and, for $C = \langle f \rangle$, where $f : S^m \to S_V$ is a p-eigenmap, the *l-th canonical coefficient of* f.

Summarizing, the l-th canonical coefficient of $C \in \mathcal{E}^p$ is a traceless symmetric bilinear map $h^l : \mathbf{R}^{m+1} \times \mathbf{R}^{m+1} \to \mathcal{H}^{2l}$, $l = 1, \ldots, p-1$, which can either be thought of as a traceless symmetric $(m+1) \times (m+1)$-matrix with entries in $\mathcal{H}^{2l}$ or as an element $h^l \in \mathcal{H}^{2l} \otimes \mathcal{H}^2$. The importance of the l-th coefficient lies in the fact that the nonvanishing of h^l means that $V^{(2l,2,0,\ldots,0)}$ does not participate in $\mathcal{F}^p$.

3.3 Conformal Fields and Raising and Lowering the Degree

Theorem 3.1.6 asserts that, for $m \geq 3$ and $p \geq 4$, $V^{(2l,2,0,\ldots,0)}$, $l = 1, \ldots, p-1$, are not components of $\mathcal{F}^p$. The direct proof of this is difficult. We therefore reduced this task to show (3.2.19). In this section we further reduce the problem to prove

$$\Psi_p|_{V^{(2(p-1),2,0,\ldots,0)}} \neq 0. \tag{3.3.1}$$

We will accomplish this by studying how degree-raising and -lowering affect homothety.

Theorem 3.3.1. *For $C \in \mathcal{E}^p$, we have*

$$\Psi_{p+1}(\Phi_p^+(C))(a,b) = \frac{\lambda_4}{4}\frac{p}{\lambda_p}\Psi_p(C)(a,b)\rho^2 + \frac{p^2}{\lambda_p\lambda_{2p}}\triangle(\Psi_p(C)(a,b))\rho^4 \tag{3.3.2}$$

and

$$\triangle(\Psi_p(C)(a,b)) = \lambda_{2p}\Psi_{p-1}(\Phi_p^-(C))(a,b). \tag{3.3.3}$$

Remark. Theorem 3.3.1 should be compared to Theorem 2.5.6 and Remark 1 at the end of Section 2.5. For perfect analogy, notice that the coefficient $m-1$ in (2.5.3) is $\lim_{q\to 0}(\lambda_q/q)$.

In (3.3.2)-(3.3.3), $\Psi_p(C)(a,b) \in \mathcal{P}^{2(p-1)}$. Restricting to S^m, and using

$$\triangle(\Psi_p(C)(a,b)) = \left(\lambda_{2(p-1)}I - \triangle^{S^m}\right)\Psi_p(C)(a,b),$$

(3.3.2)-(3.3.3) express the fact that the diagrams

$$\begin{array}{ccc} \mathcal{E}^p & \xrightarrow{\Phi_p^\pm} & \mathcal{E}^{p\pm1} \\ \Psi_p \downarrow & & \downarrow \Psi_{p\pm1} \\ \mathcal{P}^{2(p-1)} \otimes \mathcal{H}^2 & \xrightarrow{\bar{\Phi}_p^\pm} & \mathcal{P}^{2(p-1\pm1)} \otimes \mathcal{H}^2 \end{array}$$

commute. Here

$$\bar{\Phi}_p^+ = \frac{p^2}{\lambda_p \lambda_{2p}} \left(\frac{\lambda_{p+1}\lambda_{2(p+1)}}{(p+1)^2} I - \triangle^{S^m} \right) \otimes I,$$

and

$$\bar{\Phi}_p^- = \frac{1}{\lambda_{2p}} \left(\lambda_{2(p-1)} I - \triangle^{S^m} \right) \otimes I.$$

The eigenvalues of $\triangle^{S^m}$ on $\mathcal{P}^{2(p-1)} = \sum_{j=0}^{p-1} \mathcal{H}^{2j}$ are λ_{2j}, $j = 0, \ldots, p-1$. Since $\lambda_{2j} < \lambda_{p+1}\lambda_{2(p+1)}/(p+1)^2$, we see that $\bar{\Phi}_p^+$ is injective.

Corollary 3.3.2. *Let $1 \leq l \leq p-1$. Then $V^{(2l,2,0,\ldots,0)} \not\subset \mathcal{F}^p$ iff $V^{(2l,2,0,\ldots,0)} \not\subset \mathcal{F}^q$ for (some or) all $q \geq p$.*

PROOF. Without loss of generality, we may set $q = p+1$. Assume $V^{(2l,2,0,\ldots,0)} \subset \mathcal{F}^{p+1} = \ker \Psi_{p+1}$. By (3.3.3) (for p replaced by $p+1$), we have $V^{(2l,2,0,\ldots,0)} \subset \ker(\Psi_p \circ \Phi_{p+1}^-)$ so that $V^{(2l,2,0,\ldots,0)} \subset \ker(\Psi_p \circ \Phi_{p+1}^- \circ \Phi_p^+)$. On the other hand, by Theorem 2.6.1 and Corollary 3.1.5, $\Phi_{p+1}^- \circ \Phi_p^+$ is an isomorphism on $V^{(2l,2,0,\ldots,0)}$ for $1 \leq l \leq p-1$. Thus $V^{(2l,2,0,\ldots,0)} \subset \ker \Psi_p$. The proof of the converse is analogous in the use of (3.3.2).

Corollary 3.3.3. *For $m \geq 3$ and $p \geq 3$,*

$$V^{(2,2,0,\ldots,0)} \text{ and } V^{(4,2,0,\ldots,0)}$$

are not components of $\mathcal{F}^p$.

PROOF. This is certainly true for $p = 3$ by rigidity (Theorem 3.2.4). Now apply Corollary 3.3.2.

PROOF OF THEOREM 3.3.1. We will only prove (3.3.2); the proof of (3.3.3) is analogous but technically much simpler. By the definition of $\Phi_p^+(C)$, and by Lemma 2.4.2, we have

$$\Psi_{p+1}(\Phi_p^+(C)(a,b)) = \langle (C \otimes I)((f_p^+)_* X^a)^\vee, ((f_p^+)_* X^b)^\vee \rangle. \tag{3.3.4}$$

By (3.2.2) and homogeneity:

$$((f_p^+)_* X_x^a)^\vee = X_x^a(f_p^+) = \partial_a(f_p^+) - (p+1)\frac{a^*}{\rho^2} f_p^+. \tag{3.3.5}$$

Since $C \in \mathcal{E}^p$, we have $\Phi^+(C) \in \mathcal{E}^{p+1}$ (Theorem 2.6.1). Hence

$$\langle (C \otimes I) f_p^+, f_p^+ \rangle = \langle \Phi^+(C) f_{p+1}, f_{p+1} \rangle = 0.$$

Differentiating this and using symmetry of C, we also have

$$\langle (C \otimes I)\partial_a f_p^+, f_p^+ \rangle = \langle (C \otimes I) f_p^+, \partial_a f_p^+ \rangle = 0.$$

It follows that, upon substitution into (3.3.4), the second term on the right-hand side in (3.3.5) gives no contribution to (3.3.4). Hence (3.3.4) reduces

to

$$\Psi_{p+1}(\Phi_p^+(C)(a,b)) = \langle (C\otimes I)\partial_a f_p^+, \partial_b f_p^+\rangle$$
$$= \frac{\lambda_{2p}}{2\lambda_p}\sum_{i=0}^{m}\langle \partial_a\delta_i C f_p, \partial_b\delta_i f_p\rangle.$$

It remains to work out this last sum. First we derive a formula for $\partial_a\delta_b\chi$, $a,b\in\mathbf{R}^{m+1}$, $\chi\in\mathcal{H}^p$. Taking the directional derivative ∂_a of both sides of the harmonic projection formula (2.1.8):

$$\delta_b\chi = H(b^*\chi) = b^*\chi - \frac{2p}{\lambda_{2p}}\rho^2\partial_a\chi,\ \chi\in\mathcal{H}^p,$$

and using $\partial_a b^* = \langle a,b\rangle$ and $\partial_a\rho^2 = 2a^*$, $a,b\in\mathbf{R}^{m+1}$, we have

$$\partial_a\delta_b\chi = \langle a,b\rangle\chi + b^*\partial_a\chi - \frac{4p}{\lambda_{2p}}a^*\partial_b\chi - \frac{2p}{\lambda_{2p}}\rho^2\partial_a\partial_b\chi. \tag{3.3.6}$$

For $f:\mathbf{R}^{m+1}\to V$, a harmonic p-homogeneous polynomial map, and $b^* = e_i^* = x_i$, this reduces to

$$\partial_a\delta_i f = a_i f + x_i\partial_a f - \frac{4p}{\lambda_{2p}}a^*\partial_i f - \frac{2p}{\lambda_{2p}}\rho^2\partial_i\partial_a f.$$

With this, we have

$$\sum_{i=0}^{m}\langle \partial_a\delta_i C f_p, \partial_b\delta_i f_p\rangle$$
$$= \sum_{i=0}^{m}\left\langle \left(x_i\partial_a + a_i - \frac{4p}{\lambda_{2p}}a^*\partial_i - \frac{2p}{\lambda_{2p}}\rho^2\partial_i\partial_a\right)(Cf_p),\right.$$
$$\left.\left(x_i\partial_b + b_i - \frac{4p}{\lambda_{2p}}b^*\partial_i - \frac{2p}{\lambda_{2p}}\rho^2\partial_i\partial_b\right)(f_p)\right\rangle.$$

Since $C\in\mathcal{E}^p$, we have

$$\langle Cf_p, f_p\rangle = \langle \partial_a Cf_p, f_p\rangle = \langle Cf_p, \partial_a f_p\rangle = 0,\ a\in\mathbf{R}^{m+1}, \tag{3.3.7}$$

and we see that the terms that involve $a^*\partial_i$ and $b^*\partial_i$ vanish. The sum above therefore reduces to

$$\sum_{i=0}^{m}\left\langle \left(x_i\partial_a + a_i - \frac{2p}{\lambda_{2p}}\rho^2\partial_i\partial_a\right)(Cf_p), \left(x_i\partial_b + b_i - \frac{2p}{\lambda_{2p}}\rho^2\partial_i\partial_b\right)(f_p)\right\rangle.$$

Expanding, using (3.3.7) and homogeneity, we finally arrive at

$$\sum_{i=0}^{m}\langle \partial_a\delta_i C f_p, \partial_b\delta_i f_p\rangle = \left(1 - \frac{2p(p-1)}{\lambda_{2p}} + \frac{4p}{\lambda_{2p}}\right)\Psi(f)(a,b)\rho^2$$
$$+\frac{2p^2}{\lambda_{2p}^2}\triangle\Psi(f)(a,b)\rho^4.$$

(3.3.2) follows.

Let $C \in \mathcal{E}^p$ and assume that $\Psi_{p+1}(\Phi_p^+(C)) = 0$. Then (3.3.2) gives

$$\frac{\lambda_4}{4}\frac{p}{\lambda_p}\Psi_p(C)(a,b) + \frac{p^2}{\lambda_p\lambda_{2p}}\triangle(\Psi_p(C)(a,b))\rho^2 = 0.$$

The unicity part of Lemma 2.1.2 implies that $\Psi_p(C)(a,b) = 0$. The converse being obvious, we obtain that

$$\Psi_p(C) = 0 \quad \text{iff} \quad \Psi_{p+1}(\Phi_p^+(C)) = 0.$$

For $C = \langle f \rangle$, where $f : S^m \to S_V$ is a p-eigenmap, by (2.4.5), this means that degree-raising preserves homothety (and minimality).

Corollary 3.3.4. *Let $f : S^m \to S_V$ be a p-eigenmap. Then f is homothetic (with homothety λ_p/m) iff $f^+ : S^m \to S_{V\otimes\mathcal{H}^1}$ is homothetic (with homothety λ_{p+1}/m).*

This along with Theorem 2.6.1 says that applying the degree-raising operator repeatedly to a nonstandard minimal immersion, we obtain an infinite sequence of nonstandard minimal immersions of arbitrary degree.

Theorem 2.6.1 and Corollary 3.3.4 give the following:

Theorem 3.3.5. *The correspondence $f \mapsto f^+$ is injective on the congruence classes of full minimal immersions of degree p, and it gives rise to an $SO(m+1)$-equivariant imbedding $\Phi^+ : \mathcal{M}^p \to \mathcal{M}^{p+1}$.*

Degree-lowering also preserves homothety (and minimality) in the sense that if $f : S^m \to S_V$ is a full minimal immersion of degree p then $f^- : S^m \to S_{V\otimes\mathcal{H}^1}$ is also a minimal immersion of degree $p-1$. This is an immediate consequence of (3.3.3). Notice that the converse is false. Indeed, the Hopf map $Hopf : S^3 \to S^2$ is not homothetic but $Hopf^-$, being linear, must be standard, thereby homothetic.

Remark. Let $f : S^m \to S_V$ be a full quartic (degree 4) minimal immersion. By (3.3.3), $f^- : S^m \to S_{V\otimes\mathcal{H}^1}$ is a cubic minimal immersion which, by rigidity (Theorem 3.2.4), must be the standard minimal immersion $f_3 : S^m \to S_{\mathcal{H}^3}$. Comparing the ranges, we have

$$\dim(V \otimes \mathcal{H}^1) = \dim V \dim \mathcal{H}^1 \geq \dim \mathcal{H}^3.$$

Equivalently,

$$\dim V \geq \frac{\dim \mathcal{H}^3}{\dim \mathcal{H}^1} = \frac{m(m+5)}{6}, \tag{3.3.8}$$

where we used (2.1.5). We obtain that the lower bound (3.3.8) for the range V is valid for *any* full quartic minimal immersion $f : S^m \to S_V$. This is a significant improvement to Moore's linear lower bound $\dim V \geq 2m+1$ (Moore [1]) mentioned in Section 1.5. In Chapter 4 we will develop more general lower bounds for the range of minimal immersions.

We close this section by showing that the domain-dimension-raising operator Θ_m introduced in Section 2.8, also preserves homothety:

Theorem 3.3.6. *Let $f : S^m \to S_V$ be a p-eigenmap and $\tilde{f} : S^{m+1} \to S_{V\oplus(\mathcal{H}^p_{m+1}/\mathcal{H}^p_m)}$ the p-eigenmap obtained from f by raising the domain dimension. Then, for $\tilde{a}, \tilde{b} \in \mathbf{R}^{m+2}$, we have*

$$\Psi(\tilde{f})(\tilde{a},\tilde{b}) = c^2_{m,p,p}\Psi(f)(a,b),$$

where a (resp. b) is the projection of $\tilde{a}$ (resp. $\tilde{b}$) to $\mathbf{R}^{m+1} \subset \mathbf{R}^{m+2}$ and

$$c_{m,p,p} = \frac{1}{m+p-1}\frac{[m(m+2)\dots(m+2p-2)]^2}{m(m+1)\dots(m+2p-2)}.$$

In particular, f is homothetic (minimal) iff $\tilde{f}$ is.

PROOF. For the proof we need to recall from (2.8.9)-(2.8.10) that $\tilde{f}$ is obtained from a particular representation of the standard minimal immersion $f_{m+1,p} \cong \tilde{f}_{m,p}$ by replacing $f_{m,p}$ with f. By (3.2.6), we therefore have

$$\begin{aligned}\Psi(\tilde{f})(\tilde{a},\tilde{b}) &= \langle \partial_{\tilde{a}}\tilde{f}, \partial_{\tilde{b}}\tilde{f}\rangle - \langle \partial_{\tilde{a}}\tilde{f}_{m,p}, \partial_{\tilde{b}}\tilde{f}_{m,p}\rangle \\ &= c^2_{m,p,p}\Big(\langle \partial_a f, \partial_b f\rangle - \langle \partial_a f_{m,p}, \partial_b f_{m,p}\rangle\Big) \\ &= c^2_{m,p,p}\Psi(f)(a,b).\end{aligned}$$

The projections a, b of $\tilde{a}, \tilde{b}$ appear because f and $f_{m,p}$ do not depend on the variable x_{m+1}. The theorem follows.

Theorem 3.3.6 says that, using the domain-dimension-raising operator repeatedly, each full minimal immersion $f : S^m \to S^n$ of degree p gives rise to an infinite sequence of full minimal immersions

$$f^d : S^{m+d} \to S^{n+N(m+d,p)-N(m,p)}, \quad d \geq 1.$$

In particular, if f is of boundary type, $n < N(m,p)$, then so is f^d.

The analogue of Theorem 2.8.4 for the general domain-dimension-raising operator for minimal immersions is the following:

Theorem 3.3.7. *Given full minimal immersions $g_q : S^m \to S^{n_q}$ of degree q, $q = 1,\dots,p$, the map $\tilde{g} : \mathbf{R}^{m+2} \to \mathbf{R}^{N+1}$, $N = \sum_{q=1}^p (n_q+1)$, defined by (2.8.11) is a full minimal immersion of degree p.*

PROOF. We use the notations of Section 2.8, in particular, those in the proof of Theorem 2.8.4. We need to show that $\tilde{g}$ is homothetic. As before, we compare $\tilde{g}$ with $f_{m+1,p}$. Noticing that the last term in (3.2.3) does not depend on the map itself, for $a, b \in \mathbf{R}^{m+2}$, we compute

$$\begin{aligned}&\Psi(\tilde{g})(a,b) - \Psi(f_{m+1,p})(a,b) \\ &\qquad = -\langle \partial_a\tilde{g}, \partial_b\tilde{g}\rangle + \langle \partial_a f_{m+1,p}, \partial_b f_{m+1,p}\rangle\end{aligned}$$

$$= -\sum_{q=0}^{p} c_{m,p,q}^2(\langle \partial_a H(x_{m+1}^{p-q} g_q(x)), \partial_b H(x_{m+1}^{p-q} g_q(x))\rangle$$
$$-\langle \partial_a H(x_{m+1}^{p-q} f_{m,q}(x)), \partial_b H(x_{m+1}^{p-q} f_{m,q}(x))\rangle).$$

As in Section 2.8, we write $H(x_{m+1}^{p-q} g_q(x)) = K_{m,p,q}(\rho^2, x_{m+1}) g_q(x)$ (and similarly for $f_{m,q}(x)$), where $K_{m,p,q}$ is given in (2.8.3). Taking the directional derivative, each term gives two terms, for example, we have

$$\begin{aligned}\partial_a H(x_{m+1}^{p-q} g_q(x)) &= \partial_a(K_{m,p,q}(\rho^2, x_{m+1}) g_q(x)) \\ &= \partial_a(K_{m,p,q}(\rho^2, x_{m+1})) g_q(x) \\ &\quad + K_{m,p,q}(\rho^2, x_{m+1}) \partial_a g_q(x).\end{aligned}$$

Thus

$$\langle \partial_a H(x_{m+1}^{p-q} g_q(x), \partial_b H(x_{m+1}^{p-q} g_q(x)\rangle$$

splits into four terms. The term involving

$$\partial_a K_{m,p,q} \partial_b K_{m,p,q}$$

cancels with the analogous term for the standard minimal immersion. The "mixed terms" are independent of g_q since, for example,

$$\langle \partial_a g_q, g_q\rangle = \frac{1}{2}\partial_a |g_q|^2 = \frac{1}{2}\partial_a \rho^{2q}.$$

Thus they also cancel. Finally, the derivative terms, such as, $\langle \partial g_q, \partial_b g_q\rangle$ cancel with the analogous terms for the standard minimal immersion since g_q is homothetic. The theorem follows.

3.4 Exact Dimension of the Moduli $\mathcal{M}^p$

The purpose of this section is to complete the final step of the proof of Theorem 3.1.6. We begin with the following:

Lemma 3.4.1. *Let $\xi \in \mathcal{P}_{m+1}^{2(p-1)}$ have the canonical decomposition:*

$$\xi(x) = \sum_{l=0}^{p-1} \xi^l(x) \cdot \rho^{2(p-l-1)},\ \xi^l \in \mathcal{H}_m^{2l},\ l = 0, \ldots, p-1,$$

where $\rho^2 = |x|^2 = \sum_{i=0}^m x_i^2$, $x = (x_0, \ldots, x_m) \in \mathbf{R}^{m+1}$. As an element in $\mathcal{P}_{m+2}^{2(p-1)}$, we write

$$\xi(x) = \sum_{l=0}^{p-1} \tilde{\xi}^l(x, x_{m+1}) \cdot (\rho^2 + x_{m+1}^2)^{p-l-1},\ x_{m+1} \in \mathbf{R},$$

where $\tilde{\xi}_l \in \mathcal{H}_{m+1}^{2l}$, $l = 0, \ldots, p-1$. Then

$$\xi \neq 0 \quad \text{and} \quad \triangle \xi \neq 0$$

imply

$$\tilde{\xi}^{p-1} \neq 0 \quad \text{and} \quad \tilde{\xi}^{p-2} \neq 0.$$

PROOF. If $\tilde{\xi}^{p-1}$ were zero, then $\xi(x)$ would be divisible by $\rho^2 + x_{m+1}^2$. This is a contradiction since $\xi \neq 0$ and it does not depend on the variable x_{m+1}. Thus $\tilde{\xi}^{p-1} \neq 0$. Let $\triangle_m = \sum_{i=0}^{m} \partial_i^2$ and $\triangle_{m+1} = \triangle_m + \partial_{m+1}^2$. Since

$$\xi(x) \equiv \tilde{\xi}^{p-1}(x, x_{m+1}) + \tilde{\xi}^{p-2}(x, x_{m+1})(\rho^2 + x_{m+1}^2)(\bmod (\rho^2 + x_{m+1}^2)^2)$$

we have

$$\begin{aligned} \triangle_m \xi(x) &= \triangle_{m+1} \xi(x) \\ &\equiv 2(2(2p-4) + m + 1)\tilde{\xi}^{p-2}(x, x_{m+1})(\bmod (\rho^2 + x_{m+1}^2)), \end{aligned}$$

where we used (2.1.2). If $\tilde{\xi}^{p-2}$ were zero then $\triangle_m \xi$ would be a multiple of $\rho^2 + x_{m+1}^2$. This is again a contradiction since $\triangle_m \xi \neq 0$ and it does not depend on x_{m+1}.

PROOF OF THEOREM 3.1.6. By rigidity (Theorem 3.2.4), $V_{m+1}^{(2,2,0,\ldots,0)} \not\subset \mathcal{F}_m^2$, $m \geq 3$, and by degree-raising (Corollary 3.3.2), $V_{m+1}^{(2,2,0,\ldots,0)} \not\subset \mathcal{F}_m^p$, $m \geq 3$ and $p \geq 2$. Since $\mathcal{F}^p = \ker \Psi$, for any nonstandard p-eigenmap $f : S^m \to S_V$ with $\langle f \rangle \in V_{m+1}^{(2,2,0,\ldots,0)}$, we have $\Psi(f)(a,b) \neq 0$, for some $a, b \in \mathbf{R}^{m+1}$. Raising the domain dimension we obtain a nonstandard p-eigenmap $\tilde{f} : S^{m+1} \to S_{V \oplus (\mathcal{H}_{m+1}^p / \mathcal{H}_m^p)}$. Applying Lemma 3.4.1 to the polynomial $\xi = \Psi(f)(a,b)$, we see that ξ, *viewed as a polynomial in* $m+2$ *variables*, has a *nonvanishing* top canonical coefficient in $\mathcal{H}_{m+1}^{2(p-1)}$. The same holds for $\tilde{\xi} = \Psi(\tilde{f})(a,b)$ since it is a constant multiple of ξ (Theorem 3.3.6). Thus, by Theorem 3.2.7, $V_{m+2}^{(2(p-1),2,0,\ldots,0)}$ does not belong to $\mathcal{F}_{m+1}^p$. Finally, again by raising the degree (Corollary 3.3.2), $V_{m+2}^{(2(p-1),2,0,\ldots,0)} \not\subset \mathcal{F}_{m+1}^q$, $q \geq p$. In this argument $m \geq 3$ and $p \geq 2$ so that we proved Theorem 3.1.6 for $m \geq 4$ and $p \geq 2$.

It remains to consider the case $m = 3$. We claim that $V_4^{(2(p-1),2)}$ is disjoint from $\mathcal{F}_3^p$. To do this, we show that $\Psi_p | V_4^{(2(p-1),2)} \neq 0$. Let $\langle f \rangle \in V_4^{(2(p-1),2)}$ with $f : S^3 \to S_V$ of boundary type. Assume, on the contrary, that $\Psi(\langle f \rangle) = 0$. Then $\Psi(\langle \tilde{f} \rangle) = 0$ (Theorem 3.3.6) so that, by Theorem 3.1.6 (already proved for $m \geq 4$), $\tilde{f}$ is a spherical minimal immersion, i.e. $\langle \tilde{f} \rangle \in \mathcal{F}_4^p$. Consider the domain-dimension-raising operator $\Theta_3 : \mathcal{E}_3^p \to \mathcal{E}_4^p$. We have $\Theta_3(\langle f \rangle) = \langle \tilde{f} \rangle$. On the other hand, Θ_3 must send $\langle f \rangle \in V_4^{(2(p-1),2)}$ to zero since it is a homomorphism of $SO(4)$-modules and the multiplicity of $V_4^{(2(p-1),2)}$ in $\mathcal{F}_4^p|_{SO(4)}$ is zero by the branching rule (A.3.3) (as the first coefficient of any highest weight in $\mathcal{F}_4^p$ is $\leq 2(p-2)$, cf. (3.1.9)). We obtain that $\langle \tilde{f} \rangle = 0$; a contradiction. The claim follows. Finally, degree-raising (Corollary 3.3.2) implies that $V_4^{(2l,2)}$, $l = 1, \ldots, p-1$, are all disjoint from $\mathcal{F}_3^p$.

Remark. The final step in the proof of Theorem 3.1.6 is, for the most part, new. The first proof in Toth [4] is based on explicit constructions of eigenmaps as in Problems 3.5-3.8.

3.5 Isotropic Minimal Immersions

In Section 2.5 we introduced the $SO(m+1)$-module homomorphism

$$\Psi^0 : S^2(\mathcal{H}^p) \to \mathcal{P}^{2p}$$

with kernel $\mathcal{E}^p$, the linear span of the standard moduli space $\mathcal{L}^p$ parametrizing the congruence classes of full p-eigenmaps $f : S^m \to S_V$. Evaluated on a harmonic p-homogeneous polynomial map $f : \mathbf{R}^{m+1} \to V$, $\Psi^0(f)$ tells how far f is from being spherical. Using induction with respect to p along with the degree-raising and -lowering operators, in Theorem 2.5.3 we showed that Ψ^0 is onto. We reformulated this by saying that each irreducible component $V^{(2l,0,\ldots,0)} = \mathcal{H}^{2l}$ of $\mathcal{P}^{2p}$, $l = 0, \ldots, p$, is also a component in $S^2(\mathcal{H}^p)$ on which Ψ^0 is an isomorphism. This gave the isomorphism $\mathcal{E}^p \cong S^2(\mathcal{H}^p)/\mathcal{P}^{2p}$ (Theorem 2.5.1).

In Section 3.2 we introduced the $SO(m+1)$-module homomorphism

$$\Psi : \mathcal{E}^p \to \mathcal{P}^{2(p-1)} \otimes \mathcal{H}^2$$

with kernel $\mathcal{F}^p$, the linear span of the standard moduli space $\mathcal{M}^p$ parametrizing the set of congruence classes of full minimal immersions $f : S^m \to S_V$ of degree p. Evaluated on a p-eigenmap $f : S^m \to S_V$, $\Psi(f)$ tells how far f is from being homothetic. Using induction with respect to p along with the degree-raising and -lowering operators, in Sections 3.3-3.4, we then showed that the irreducible components common in $\mathcal{E}^p$ and $\mathcal{P}^{2(p-1)} \otimes \mathcal{H}^2$ are $V^{(2l,2,0,\ldots,0)}$, $l = 1, \ldots, p-1$, and that Ψ, restricted to each of these, is an isomorphism. This led to the exact determination of the $SO(m+1)$-module structure of $\mathcal{F}^p$ (Theorem 3.1.6).

The analogy between Ψ^0 and Ψ suggests that there may be higher order versions of these constructions. The purpose of the present section is to give a detailed account of this extension.

Let $f : S^m \to S_V$ be a minimal immersion of degree p. Let $\beta_k(f)$, $k = 1, \ldots, p$, denote the *k-th fundamental form* of f (Appendix 2). $\beta_k(f)$ are defined on a (maximal) open dense subset $D_f \subset S^m$. For $0 \leq k \leq p$, we introduce the $(2k)$-tensor $\Psi_p^k(f)$ on $D_f \subset S^m$ by

$$\begin{aligned} &\Psi_p^k(f)(X_1, \ldots, X_{2k}) \qquad (3.5.1)\\ &\quad = \langle \beta_k(f)(X_1, \ldots, X_k), \beta_k(f)(X_{k+1}, \ldots, X_{2k}) \rangle \\ &\qquad - \langle \beta_k(f_p)(X_1, \ldots, X_k), \beta_k(f_p)(X_{k+1}, \ldots, X_{2k}) \rangle, \end{aligned}$$

where $X_1, \ldots, X_{2k} \in T(S^m)$. Since $\beta(f) = f_*$, for $k = 1$ this definition of Ψ_p^1 agrees with that of Ψ_p in (3.2.1) for p-eigenmaps $f : S^m \to S_V$.

Moreover, for a harmonic p-homogeneous polynomial map $f : \mathbf{R}^{m+1} \to V$, we have $\beta_0(f) = f$, so that (3.5.1), for $k = 0$, agrees with the definition of Ψ^0 given in Section 2.5.

A minimal immersion $f : S^m \to S_V$ of degree p is said to be *isotropic of order* k, $2 \leq k \leq p$, if

$$\Psi_p^2(f) = \ldots = \Psi_p^k(f) = 0. \tag{3.5.2}$$

Using the extended meaning of the fundamental forms for $k = 0, 1$, we see that a harmonic p-homogeneous polynomial map $f : \mathbf{R}^{m+1} \to V$ is isotropic of order 0 iff it is spherical; and a p-eigenmap $f : S^m \to S_V$ is isotropic of order 1 iff it is a minimal immersion of degree p. Thus, the order of isotropy (in the appropriate category) is defined for all $0 \leq k \leq p$.

We restrict ourselves to conformal vector fields X^a, $a \in \mathbf{R}^{m+1}$ (Section 3.2). Let $f : S^m \to S_V$ be any minimal immersion of degree p. We claim that

$$\beta_k(f)(X^{a_1}, \ldots, X^{a_k})^{\check{}} = \partial_{a_1} \ldots \partial_{a_k} f + \text{ lower order terms},$$
$$a_1, \ldots, a_k \in \mathbf{R}^{m+1},\ k \geq 1. \tag{3.5.3}$$

As before, $\check{}$ means translating vectors to the origin. We prove (3.5.3) by induction with respect to k. The basic step, $k = 1$, follows from the definitions:

$$\beta_1(f)(X^a)^{\check{}} = f_*(X^a)^{\check{}} = X^a(f) = \partial_a f - p\frac{a^*}{\rho^2} f, \quad \rho^2 = |x|^2,$$

where we used the extension (3.2.2) of X^a. For the general induction step $k - 1 \Rightarrow k$, we recall from the definition of higher fundamental forms that, up to the orthogonal projection to the previous osculating subbundle that, by the induction hypothesis, is obtained by subtracting an appropriate multiple of lower order terms, $\beta_k(f)(X^{a_1}, \ldots, X^{a_k})^{\check{}}$ is

$$(\nabla_{X^{a_k}} \beta_{k-1}(f))(X^{a_1}, \ldots, X^{a_{k-1}})^{\check{}}.$$

The principal term of this is the same as the principal term of

$$X^{a_k}(\beta_{k-1}(X^{a_1}, \ldots, X^{a_{k-1}})^{\check{}}).$$

The principal term of X^{a_k} is ∂_{a_k} and, by the induction hypothesis, the principal term of $\beta_{k-1}(f)(X^{a_1}, \ldots, X^{a_{k-1}})^{\check{}}$ is $\partial_{a_1} \ldots \partial_{a_{k-1}} f$. (3.5.3) follows.

In $\Psi_p^k(f)(X^{a_1}, \ldots, X^{a_{2k}})$, $a_1, \ldots, a_{2k} \in \mathbf{R}^{m+1}$, we retain only the principal term in (3.5.3) for the k-th fundamental forms of f and f_p. The difference of the scalar products in (3.5.1) gives

$$\langle \partial_{a_1} \ldots \partial_{a_k} f, \partial_{a_{k+1}} \ldots \partial_{a_{2k}} f \rangle - \langle \partial_{a_1} \ldots \partial_{a_k} f_p, \partial_{a_{k+1}} \ldots \partial_{a_{2k}} f_p \rangle$$
$$= \langle \partial_{a_1} \ldots \partial_{a_k} \langle f \rangle f_p, \partial_{a_{k+1}} \ldots \partial_{a_{2k}} f_p \rangle. \tag{3.5.4}$$

To generalize this, for $C \in S^2(\mathcal{H}^p)$ and $0 \leq k \leq p$, *we define*

$$\Psi_p^k(C)(a_1, \ldots, a_{2k}) = \langle \partial_{a_1} \ldots \partial_{a_k} C f_p, \partial_{a_{k+1}} \ldots \partial_{a_{2k}} f_p \rangle, \tag{3.5.5}$$

for $a_1, \ldots, a_{2k} \in \mathbf{R}^{m+1}$. Clearly, this is a $2(p-k)$-homogeneous polynomial on $\mathbf{R}^{m+1}$. Our immediate purpose is to show that the deletion of the lower order terms does not affect the difference of the scalar products in (3.5.1), i.e. we have

$$\Psi_p^k(f)(X^{a_1}, \ldots, X^{a_{2l}}) = \Psi_p^k(\langle f \rangle)(a_1, \ldots, a_{2k})$$

for any p-eigenmap $f : S^m \to S_V$, isotropic of order $k-1$.

In what follows we use standard multiindex notation; for $I = \{i_1, \ldots, i_l\}$, we set $|I| = l$ and $\partial_{a_I} = \partial_{a_1} \ldots \partial_{a_l}$.

Lemma 3.5.1. *Given $C \in S^2(\mathcal{H}^p)$ and $1 \leq k \leq p$, assume that*

$$\Psi_p^0(C) = \Psi_p^1(C) = \ldots = \Psi_p^{k-1}(C) = 0.$$

Then

$$\langle \partial_{a_I} C f_p, \partial_{a_J} f_p \rangle = 0 \tag{3.5.6}$$

for all I and J with $0 \leq |I| + |J| \leq 2k-1$, $I, J \subset \{1, \ldots, 2k\}$; and

$$\Psi_p^k(C)(a_1, \ldots, a_{2k}) = (-1)^{(|I|-|J|)/2} \langle \partial_{a_I} C f_p, \partial_{a_J} f_p \rangle \tag{3.5.7}$$

for all I and J disjoint with $I \cup J = \{1, \ldots, 2k\}$.

PROOF. We proceed by induction with respect to k. For $k = 1$, the assumption reduces to

$$\Psi_p^0(C) = \langle C f_p, f_p \rangle = 0.$$

Differentiating, we obtain

$$\langle \partial_a C f_p, f_p \rangle = \langle C f_p, \partial_a f_p \rangle = 0, \quad a \in \mathbf{R}^{m+1},$$

and (3.5.6) follows. Differentiating this again:

$$\Psi_p^1(C)(a, b) = \langle \partial_a C f_p, \partial_b f_p \rangle = -\langle \partial_a \partial_b C f_p, f_p \rangle, \quad a, b \in \mathbf{R}^{m+1},$$

and we get (3.5.7).

For the general induction step $k-1 \Rightarrow k$, it will be convenient to introduce the notation

$$F(I; J) = F(i_1, \ldots, i_r; i_{r+1}, \ldots, i_{r+s}) = \langle \partial_{a_I} C f_p, \partial_{a_J} f_p \rangle,$$

where $I = \{i_1, \ldots, i_r\}$ and $J = \{i_{r+1}, \ldots, i_{r+s}\}$. Symmetry of C implies $F(I; J) = F(J; I)$, in particular, we can write $F(I; \emptyset) = F(I)$ and $F(\emptyset; J) = F(J)$. With this notation, the induction hypothesis writes as

$$F(i_1, \ldots, i_r; i_{r+1}, \ldots, i_{r+s}) = 0, \ 0 \leq r + s \leq 2k-2.$$

In particular,

$$F(i_1, \ldots, i_r; i_{r+1}, \ldots, i_{2k-2}) = 0, \ 0 \leq r \leq 2k-2.$$

Differentiating this by $\partial_{a_{2k-1}}$, we get

$$F(i_1,\dots,i_r;i_{r+1},\dots,i_{2k-2},i_{2k-1}) \tag{3.5.8}$$
$$= -F(i_1,\dots,i_r,i_{2k-1};i_{r+1},\dots,i_{2k-2}), \quad 0 \le r \le 2k-2.$$

Using this repeatedly, we obtain

$$F(i_1,\dots,i_{k-1};i_k,\dots,i_{2k-1}) = (-1)^k F(i_1,\dots,i_{2k-1}),$$
$$F(i_k,\dots,i_{2k-1};i_1,\dots,i_{k-1}) = (-1)^{k-1} F(i_1,\dots,i_{2k-1}).$$

By symmetry, these two terms are equal, and so $F(i_1,\dots,i_{2k-1}) = 0$. Using (3.5.8) repeatedly again, we obtain

$$F(i_1,\dots,i_r;i_{r+1},\dots,i_{2k-1}) = 0,\ 0 \le r \le 2k-1. \tag{3.5.9}$$

This is (3.5.6). Now (3.5.7) is obtained by differentiating (3.5.6).

Remark. In Lemma 3.5.1, it is enough to assume that $\Psi_p^{k-1}(C) = 0$. This is because homogeneity in (3.5.5) allows the removal of the derivatives.

Theorem 3.5.2. *Let $f : S^m \to S_V$ be a p-eigenmap and assume that f is isotropic of order $k-1$, $k \ge 1$. Then, for $a_1,\dots,a_{2k} \in \mathbf{R}^{m+1}$, we have*

$$\Psi_p^k(f)(X^{a_1},\dots,X^{a_{2k}}) = \Psi_p^k(\langle f\rangle)(a_1,\dots,a_{2k}). \tag{3.5.10}$$

PROOF. We proceed by induction with respect to k. The basic step ($k = 1$) is the statement of Lemma 3.2.2. Now let $k \ge 2$ and consider the general induction step $0,\dots,k-1 \Rightarrow k$. Assume that f is isotropic of order $k-1$, i.e.

$$\Psi_p^0(f) = \dots = \Psi_p^{k-1}(f) = 0. \tag{3.5.11}$$

Since f is also isotropic of order $\le k-2$, the induction hypothesis implies

$$\Psi_p^l(f)(X^{a_1},\dots,X^{a_{2l}}) = \Psi_p^l(\langle f\rangle)(a_1,\dots,a_{2l}) = 0,\ 0 \le l \le k-1. \tag{3.5.12}$$

(3.5.11) and (3.5.12) together give

$$\Psi_p^l(\langle f\rangle)(a_1,\dots,a_{2l}) = 0,\ 0 \le l \le k-1. \tag{3.5.13}$$

This is precisely the assumption in Lemma 3.5.1. Using the expansion in (3.5.3) for f and f_p, (3.5.1) can be written as $\Psi_p^k(\langle f\rangle)(a_1,\dots,a_{2k})$ plus a sum of terms of type

$$\langle \partial_{a_I} f, \partial_{a_J} f\rangle - \langle \partial_{a_I} f_p, \partial_{a_J} f_p\rangle = \langle \partial_{a_I} C f_p, \partial_{a_J} f_p\rangle,$$

where I and J satisfy $|I| + |J| \le 2k-1$. By Lemma 3.5.1, these all vanish. Theorem 3.5.2 follows.

Corollary 3.5.3. *Let $f : S^m \to S_V$ be a p-eigenmap. Then f is isotropic of order k iff*

$$\Psi_p^0(\langle f\rangle) = \dots = \Psi_p^k(\langle f\rangle) = 0. \tag{3.5.14}$$

PROOF. We need only to prove that (3.5.14) implies isotropy of order k (Theorem 3.5.2). Since the conformal fields span each tangent space, it is enough to prove that

$$\Psi_p^l(f)(X^{a_1}, \ldots, X^{a_{2l}}) = 0, \ a_1, \ldots, a_{2l} \in \mathbf{R}^{m+1}, \ 0 \leq l \leq k.$$

Notice now that, in the proof of Theorem 3.5.2, in deriving (3.5.10) we used only (3.5.13) which is precisely (3.5.14) (with k replaced by $k+1$). The corollary follows.

The extension of Theorem 3.2.4 (rigidity) is the following:

Theorem 3.5.4. *Let $f : S^m \to S_V$ be a minimal immersion of degree p and order of isotropy k. If $p \leq 2k+1$ then f is congruent to f_p.*

PROOF. Let $C = \langle f \rangle$. According to Corollary 3.5.3:

$$\Psi_p^0(C) = \ldots = \Psi_p^k(C) = 0.$$

Let $p \leq 2k+1$. By (3.5.6) (with k replaced by $k+1$), we have

$$\langle \partial_{a_1} \ldots \partial_{a_p} C f_p, f_p \rangle = 0, \ a_1, \ldots, a_p \in \mathbf{R}^{m+1}.$$

$\partial_{a_1} \ldots \partial_{a_p} C f_p$ is constant since f_p is of degree p. Since f_p is full we obtain

$$\partial_{a_1} \ldots \partial_{a_p} C f_p = 0.$$

By homogeneity, we can remove $\partial_{a_1} \ldots \partial_{a_p}$ and arrive at

$$C f_p = 0.$$

Again by fullness, this implies $C = \langle f \rangle = 0$. Hence f is standard.

Since (3.5.14) is a linear condition, Corollary 3.5.3 implies that the set of congruence classes of full minimal immersions $f : S^m \to S_V$ of degree p and order of isotropy k (≥ 1) is parametrized by a linear slice $\mathcal{M}^{p;k}$ of $\mathcal{M}^p$. The linear span $\mathcal{F}^{p;k} \subset \mathcal{F}^p$ of $\mathcal{M}^{p;k}$ is an $SO(m+1)$-submodule. (This follows since isotropy is preserved under precomposition by isometries on the domain, or equivalently, because Ψ_p^k is a homomorphism of $SO(m+1)$-modules.)

The main result of this section describes the $SO(m+1)$-module structure of $\mathcal{F}^{p;k}$.

Theorem 3.5.5. *Let $\mathcal{F}^{p;k} \subset S^2(\mathcal{H}^p)$ denote the $SO(m+1)$-module, the linear span of the space $\mathcal{M}^{p;k}$ parametrizing the set of congruence classes of full minimal immersions $f : S^m \to S_V$ of degree p and order of isotropy k. Then $\mathcal{M}^{p;k}$ is trivial for $p \leq 2k+1$. For $m \geq 3$ and $p \geq 2(k+1)$, $\mathcal{M}^{p;k}$ is a compact convex body in $\mathcal{F}^{p;k}$, and we have*

$$\mathcal{F}^{p;k} = \sum_{(u,v) \in \triangle_{k+1}^p;\, u,v \text{ even}} V^{(u,v,0,\ldots,0)}, \tag{3.5.15}$$

where $\triangle_{k+1}^p$ is the triangle with vertices $(2(k+1), 2(k+1))$, (p,p) and $(2(p-k-1), 2(k+1))$.

Remark. Weingart [1] recently gave a new algebraic proof of Theorem 3.5.5. In addition, he also proved the following: Let M be a compact analytic Riemannian manifold, and $f_1 : M \to S_{V_1}$ and $f_2 : M \to S_{V_2}$ isometric minimal immersions of (geometric) degree p. Assume that, for $1 \leq l \leq [p/2]$, we have

$$\begin{aligned}&\langle \beta_l(f_1)(X_1, \ldots, X_l), \beta_l(f_1)(X_{l+1}, \ldots, X_{2l})\rangle \\ &\quad = \langle \beta_l(f_2)(X_1, \ldots, X_l), \beta_l(f_2)(X_{l+1}, \ldots, X_{2l})\rangle,\end{aligned}$$

for $X_1, \ldots, X_{2l} \in T(U)$, where $U \subset M$ is an open set. Then f_1 and f_2 are congruent.

The first statement of Theorem 3.5.5 is a reformulation of Theorem 3.5.4, just proved. For the second statement we will proceed by induction with respect to the order of isotropy. We let $p \geq 2(k+1)$ and assume that the set of congruence classes of full minimal immersions $f : S^m \to S_V$ of degree p and order of isotropy $k-1$ are parametrized by

$$\mathcal{M}^{p;k-1} = \{C \in \mathcal{F}^{p;k-1} \,|\, C + I \geq 0\}$$

with linear span $\mathcal{F}^{p;k-1}$. This latter assumption amounts geometrically to

$$\mathcal{F}^{p;k-1} = \{C \in S^2(\mathcal{H}^p) \,|\, \Psi_p^0(C) = \ldots = \Psi_p^{k-1}(C) = 0\} \tag{3.5.16}$$

and algebraically to

$$\mathcal{F}^{p;k-1} = \sum_{(u,v)\in\triangle_k^p\,;\,u,v\,\text{even}} V^{(u,v,0,\ldots,0)}. \tag{3.5.17}$$

Given $C \in \mathcal{F}^{p,k-1}$, (3.5.7), for $J = \emptyset$, implies that

$$\Psi_p^k(C)(a_1, \ldots, a_{2k}) = \langle \partial_{a_1} \ldots \partial_{a_{2k}} C f_p, f_p\rangle$$

is symmetric in all vectorial variables. Because of this symmetry, we will consider

$$\Psi_p^k(C) : S^{2k}(\mathbf{R}^{m+1}) \to \mathcal{P}^{2(p-k)}$$

as a linear map or as an element

$$\Psi_p^k(C) \in \mathcal{P}^{2(p-k)} \otimes \mathcal{P}^{2k},$$

where $\mathcal{P}^{2k} = S^{2k}(\mathcal{H}^1)$, $\mathcal{H}^1 = (\mathbf{R}^{m+1})^*$. Harmonicity of f_p implies that contraction of $\Psi_p^k(C)(a_1, \ldots, a_{2k})$ with respect to any two vectorial variables is zero. This means that the component of $\Psi_p^k(C)$ in $\mathcal{P}^{2(p-k)} \otimes \mathcal{P}^{2(k-1)} \cdot \rho^2$ is zero. By (2.1.1), the orthogonal complement of $\mathcal{P}^{2(k-1)} \cdot \rho^2$ in $\mathcal{P}^{2k}$ is $\mathcal{H}^{2k}$. We obtain that $\Psi_p^k(C) \in \mathcal{P}^{2(p-k)} \otimes \mathcal{H}^{2k}$. Varying $C \in \mathcal{F}^{p;k-1}$ we end up with the linear map

$$\Psi_p^k : \mathcal{F}^{p;k-1} \to \mathcal{P}^{2(p-k)} \otimes \mathcal{H}^{2k}. \tag{3.5.18}$$

This is a homomorphism of $SO(m+1)$-modules since

$$\Psi_p^k(g \cdot C)(g \cdot a_1, \ldots, g \cdot a_{2k}) = \Psi_p^k(C)(a_1, \ldots, a_{2k}) \circ g^{-1},$$

for $g \in SO(m+1)$ and $a_1, \ldots, a_{2k} \in \mathbf{R}^{m+1}$. (The proof is the same as for Lemma 3.2.6.)

Let $f : S^m \to S_V$ be a minimal immersion of degree p and order of isotropy $k-1$. Then $\langle f \rangle \in \mathcal{M}^{p;k-1}$ (induction hypothesis) and by Corollary 3.5.3, $\Psi_p^k(\langle f \rangle)(a_1, \ldots, a_{2k})$ is zero for all $a_1, \ldots, a_{2k} \in \mathbf{R}^{m+1}$ iff f is isotropic of order k. We obtain that the linear slice

$$\bar{\mathcal{M}}^{p;k} = \mathcal{M}^{p;k-1} \cap \ker \Psi_p^k$$

parametrizes the set of congruence classes of full minimal immersions $f : S^m \to S_V$ of degree p and order of isotropy k. We set

$$\bar{\mathcal{F}}^{p;k} = \ker \Psi_p^k. \tag{3.5.19}$$

The general induction step and, hence, the proof of Theorem 3.5.5 will be accomplished if we show that $\bar{\mathcal{F}}^{p;k} = \mathcal{F}^{p;k}$ with $\mathcal{F}^{p;k}$ defined in (3.5.15). In (3.5.18) we consider the canonical decomposition

$$\mathcal{P}^{2(p-k)} = \sum_{l=0}^{p-k} \mathcal{H}^{2l} \cdot \rho^{2(p-k-l)}.$$

As $SO(m+1)$-modules, we have

$$\mathcal{P}^{2(p-k)} \otimes \mathcal{H}^{2k} = \sum_{l=0}^{p-k} \mathcal{H}^{2l} \otimes \mathcal{H}^{2k}. \tag{3.5.20}$$

Decomposing the tensor products of spherical harmonics according to (3.1.7), we see that the only irreducible $SO(m+1)$-modules that are contained in both $\mathcal{F}^{p;k-1}$ and $\mathcal{P}^{2(p-k)} \otimes \mathcal{H}^{2k}$ are

$$V^{(2l,2k,0,\ldots,0)}, \ l = k, \ldots, p-k. \tag{3.5.21}$$

These are precisely the modules that correspond to the base of the triangle $\triangle_k^p$. In particular, since they are disjoint from $\mathcal{F}^{p;k}$, we obtain

$$\mathcal{F}^{p;k} \subset \bar{\mathcal{F}}^{p;k}.$$

Equality holds iff

$$V^{(2l,2k,0,\ldots,0)} \not\subset \mathcal{F}^{p;k}, \ l = k, \ldots, p-k.$$

By (3.5.19), this is guaranteed by

$$\Psi_p^k |_{V^{(2l,2k,0,\ldots,0)}} \neq 0, \ l = k, \ldots, p-k. \tag{3.5.22}$$

As before ($k = 1$) our immediate purpose is to reduce this to

$$\Psi_p^k |_{V^{(2(p-k),2k,0,\ldots,0)}} \neq 0, \ p \geq 2(k+1), \tag{3.5.23}$$

i.e. nonvanishing at the southeast vertex $(2(p-k), 2k)$ of $\triangle_k^p$. The key to this is to show that degree-raising and -lowering preserve isotropy. The following result is a common generalization of Theorem 2.5.6 ($k = 0$) and Theorem 3.3.1 ($k = 1$).

Theorem 3.5.6. *Let $C \in S^2(\mathcal{H}^p)$ and assume that*

$$\Psi_p^0(C) = \ldots = \Psi_p^{k-1}(C) = 0.$$

Then, for $a_1, \ldots, a_{2k} \in \mathbf{R}^{m+1}$, we have

$$\Psi_{p+1}^k(\Phi_p^+(C))(a_1, \ldots, a_{2k}) = \frac{\lambda_{4k}}{4k}\frac{p}{\lambda_p}\Psi_p^k(C)(a_1, \ldots, a_{2k})\rho^2$$
$$+\frac{p^2}{\lambda_p\lambda_{2p}}\triangle(\Psi_p^k(C)(a_1, \ldots, a_{2k}))\rho^4, \tag{3.5.24}$$

and

$$\triangle(\Psi_p^k(C)(a_1, \ldots, a_{2k})) = \lambda_{2p}\Psi_{p-1}^k(\Phi_p^-(C))(a_1, \ldots, a_{2k}). \tag{3.5.25}$$

PROOF. As in the proof of Theorem 3.3.1, we compute

$$\Psi_{p+1}^k(\Phi_p^+(C))(a_1, \ldots, a_{2k}) \tag{3.5.26}$$
$$= \frac{\lambda_{2p}}{2\lambda_p}\sum_{i=0}^{m}\langle \partial_{a_1}\ldots\partial_{a_k}\delta_i C f_p, \partial_{a_{k+1}}\ldots\partial_{a_{2k}}\delta_i f_p\rangle.$$

To work out the sum on the right-hand side, we use the product rule in differentiating $H(a^*\chi)$, $\chi \in \mathcal{H}^p$, $a \in \mathbf{R}^{m+1}$ in (2.1.8). We obtain

$$\partial_{a_1}\ldots\partial_{a_k}\delta_a\chi = \partial_{a_1}\ldots\partial_{a_k}\left(a^*\chi - \frac{2p}{\lambda_{2p}}\rho^2\partial_a\chi\right)$$
$$= \sum_{l=1}^{k}\langle a_l, a\rangle\partial_{a\widehat{{}_l}}\chi + a^*\partial_{a_1}\ldots\partial_{a_k}\chi \tag{3.5.27}$$
$$-\frac{4p}{\lambda_{2p}}\sum_{l=1}^{k}a_l^*\partial_a\partial_{a\widehat{{}_l}}\chi - \frac{4p}{\lambda_{2p}}\sum_{1\le l<l'\le k}\langle a_l, a_{l'}\rangle\partial_{a\widehat{{}_{ll'}}}\chi$$
$$-\frac{2p}{\lambda_{2p}}\rho^2\partial_a\partial_{a_1}\ldots\partial_{a_k}\chi,$$

where

$$\partial_{a\widehat{{}_l}} = \partial_{a_1}\ldots\widehat{\partial_{a_l}}\ldots\partial_{a_k}, 1 \le l \le k,$$

and

$$\partial_{a\widehat{{}_{ll'}}} = \partial_{a_1}\ldots\widehat{\partial_{a_l}}\ldots\widehat{\partial_{a_{l'}}}\ldots\partial_{a_k}, 1 \le l, l' \le k,$$

and $\widehat{\ }$ means that the corresponding factor is absent. We need to substitute $\chi = Cf_p$ and $\chi = f_p$ in (3.5.27), and take the scalar product as in (3.5.26). Using (3.5.6) and its Laplacian, we have

$$\sum_{i=0}^{m}\langle \partial_i\partial_{a_I}Cf_p, \partial_i\partial_{a_J}f_p\rangle = 0, \ |I| + |J| \le 2k-1. \tag{3.5.28}$$

We see that the third and fourth terms in (3.5.27) do not contribute any terms to (3.5.26). Writing out only the remaining terms, we have

$$\begin{aligned}
&\sum_{i=0}^{m}\langle\partial_{a_1}\dots\partial_{a_k}\delta_i C f_p,\partial_{a_{k+1}}\dots\partial_{a_{2k}}\delta_i f_p\rangle\\
&=\sum_{i=0}^{m}\left\langle\left(\sum_{l=1}^{k}\langle a_l,e_i\rangle\partial_{\widehat{a_l}}+x_i\partial_{a_1}\dots\partial_{a_k}-\frac{2p}{\lambda_{2p}}\rho^2\partial_i\partial_{a_1}\dots\partial_{a_k}\right)(Cf_p),\right.\\
&\left.\left(\sum_{l=k+1}^{2k}\langle a_l,e_i\rangle\partial_{\widehat{a_l}}+x_i\partial_{a_{k+1}}\dots\partial_{a_{2k}}-\frac{2p}{\lambda_{2p}}\rho^2\partial_i\partial_{a_{k+1}}\dots\partial_{a_{2k}}\right)(f_p)\right\rangle\\
&=-\frac{2p}{\lambda_{2p}}\sum_{l=1}^{k}\langle\partial_{\widehat{a_l}}Cf_p,\partial_{a_l}\partial_{a_{k+1}}\dots\partial_{a_{2k}}f_p\rangle\rho^2\\
&\quad-\frac{2p}{\lambda_{2p}}\sum_{l=k+1}^{2k}\langle\partial_{\widehat{a_l}}f_p,\partial_{a_l}\partial_{a_1}\dots\partial_{a_k}Cf_p\rangle\rho^2\\
&\quad+\langle\partial_{a_1}\dots\partial_{a_k}Cf_p,\partial_{a_{k+1}}\dots\partial_{a_{2k}}f_p\rangle\rho^2\\
&\quad-\frac{4p(p-k)}{\lambda_{2p}}\langle\partial_{a_1}\dots\partial_{a_k}Cf_p,\partial_{a_{k+1}}\dots\partial_{a_{2k}}f_p\rangle\rho^2\\
&\quad+\frac{2p^2}{\lambda_{2p}^2}\triangle\langle\partial_{a_1}\dots\partial_{a_k}Cf_p,\partial_{a_{k+1}}\dots\partial_{a_{2k}}f_p\rangle\rho^4.
\end{aligned}$$

Here $a_{\widehat{l}}$, for $k+1\le l\le 2k$, has the obvious meaning. Putting these together, and using Lemma 3.5.1, we obtain (3.5.24). The proof of (3.5.25) is analogous and technically much simpler since it does not require differentiation of the harmonic projection formula.

Aside from the induction process in proving (3.5.15), think for a moment of $\mathcal{F}^{p;k-1}$ as being the joint kernel of $\Psi_p^0,\dots,\Psi_p^{k-1}$ on $S^2(\mathcal{H}^p)$ as in (3.5.16), and define Ψ_p^k in (3.5.18) as in the text. Then (3.5.24)-(3.5.25) can be interpreted as commutativity of the diagrams

$$\begin{array}{ccc}
\mathcal{F}^{p;k-1} & \xrightarrow{\Phi_p^{\pm}} & \mathcal{F}^{p\pm1;k-1}\\
\Psi_p^k\downarrow & & \downarrow\Psi_{p\pm1}^k\\
\mathcal{P}^{2(p-k)}\otimes\mathcal{H}^{2k} & \xrightarrow{\bar\Phi_p^{\pm}} & \mathcal{P}^{2(p-k\pm1)}\otimes\mathcal{H}^{2k}
\end{array}$$

where

$$\bar\Phi_p^{+}=\frac{p^2}{\lambda_p\lambda_{2p}}\left(\frac{\lambda_{p+k}\lambda_{2(p+k)}}{(p+k)^2}I-\triangle^{S^m}\right)\otimes I,$$

and

$$\bar\Phi_p^{-}=\frac{1}{\lambda_{2p}}\left(\lambda_{2(p-k)}I-\triangle^{S^m}\right)\otimes I.$$

Since the eigenvalues λ_{2j}, $j = 0, \ldots, p-k$, of $\triangle^{S^m}$ on $\mathcal{P}^{2(p-k)}$ satisfy $\lambda_{2j} < \lambda_{p+k}\lambda_{2(p+k)}/(p+k)^2$, we see that $\bar{\Phi}_p^+$ is injective. We now claim that this implies that $\bar{\Phi}_p^+ : S^2(\mathcal{H}^p) \to S^2(\mathcal{H}^{p+1})$ is injective. This crucial fact, used in the proof of Theorem 2.6.1, is derived from Corollary A.3.3 in Appendix 3 as a byproduct of the decomposition of $S^2(\mathcal{H}^p)$ into irreducible components. Here is an independent proof:

Assume $0 \neq C \in S^2(\mathcal{H}^p)$. By Theorem 3.5.4, $\mathcal{F}^{p;k}$ is trivial for $2k+1 \geq p$. Hence, there exists $0 \leq k \leq [(p-1)/2]$ such that $C \in \mathcal{F}^{p;k-1}$ but $C \notin \mathcal{F}^{p;k}$. Since the kernel of Ψ_p^k is $\mathcal{F}^{p;k}$, injectivity of $\bar{\Phi}_p^+$ in the diagram above implies that $\bar{\Phi}_p^+(\Psi_p^k(C)) \neq 0$. Since the diagram is commutative, we obtain $\Phi_p^+(C) \neq 0$. Hence Φ_p^+ is injective.

Corollary 3.5.7. *Let $k \leq l \leq p-k$. Then $\Psi_p^k|_{V^{(2l,2k,0,\ldots,0)}} \neq 0$ iff $\Psi_q^k|_{V^{(2l,2k,0,\ldots,0)}} \neq 0$ for (some or) all $q \geq p$.*

PROOF. We may assume that $q = p+1$. Let $\Psi_{p+1}^k|_{V^{(2l,2k,0,\ldots,0)}} = 0$. By (3.5.25), we have $\Psi_p^k \circ \Phi_{p+1}^-|_{V^{(2l,2k,0,\ldots,0)}} = 0$. By Corollary 3.1.5, Φ_{p+1}^- is an isomorphism on $V^{(2l,2k,0,\ldots,0)}$ for $k \leq l \leq p-k$, so that $\Psi_p^k|_{V^{(2l,2k,0,\ldots,0)}} = 0$. The converse is analogous in the use of (3.5.24).

Corollary 3.5.8. *For $m \geq 3$ and $p \geq 2k$, $\Psi_p^k|_{V^{(2k,2k,0,\ldots,0)}} \neq 0$.*

PROOF. By rigidity (Theorem 3.5.4), $\Psi_{2k}^k|_{V^{(2k,2k,0,\ldots,0)}} \neq 0$. Now Corollary 3.5.7 applies.

To complete the stated reduction, we now assume (3.5.23), or what is the same:

$$\Psi_{k+l}^k|_{V^{(2l,2k,0,\ldots,0)}} \neq 0, \quad l \geq k+2.$$

By Corollary 3.5.7, $k+l$ can be replaced by $p \geq k+l$ and this is precisely (3.5.22). We finally observe that (3.5.23) is equivalent to the existence of $C \in S^2(\mathcal{H}^p)$ such that

$$\Psi_p^0(C) = \ldots = \Psi_p^{k-1}(C) = 0 \tag{3.5.29}$$

and

$$H(\Psi_p^k(C)(a_1, \ldots, a_{2k})) \neq 0, \text{ for some } a_1, \ldots, a_{2k} \in \mathbf{R}^{m+1}. \tag{3.5.30}$$

In fact, taking the harmonic part of the polynomial

$$\Psi_p^k(C)(a_1, \ldots, a_{2k}),$$

we land in $\mathcal{H}^{2(p-k)}(\otimes \mathcal{H}^{2k})$ of (3.5.20) and this contains the component $V^{(2(p-k),2k,0,\ldots,0)}$ corresponding to the southeast vertex of $\triangle_k^p$.

We now bring in the domain-dimension-raising operator $\Theta_m : S^2(\mathcal{H}_m^p) \to S^2(\mathcal{H}_{m+1}^p)$, $C \mapsto C \oplus 0_{\mathcal{H}_{m+1}^p/\mathcal{H}_m^p}$ introduced in Section 2.8. The next lemma follows from the definitions.

Lemma 3.5.9. *Let $C \in S^2(\mathcal{H}_m^p)$. Then, for $0 \le k \le p$, we have*

$$\Psi_{m+1,p}^k(\Theta_m(C))(\tilde{a}_1, \ldots, \tilde{a}_{2k}) = c_{m,p,p}^2 \Psi_{m,p}^k(C)(a_1, \ldots, a_{2k}), \qquad (3.5.31)$$

where $a_r \in \mathbf{R}^{m+1}$ is the orthogonal projection of $\tilde{a}_r \in \mathbf{R}^{m+2}$ to $\mathbf{R}^{m+1}$.

PROOF OF THEOREM 3.5.5. Let $f : S^m \to S_V$ be a p-eigenmap and assume that f is isotropic of order $k-1$. By (3.5.31), $\tilde{f}$ is also isotropic of order $k-1$:

$$\Psi_{m+1,p}^0(\langle \tilde{f} \rangle) = \ldots = \Psi_{m+1,p}^{k-1}(\langle \tilde{f} \rangle) = 0,$$

and we have

$$\Psi_{m+1,p}^k(\langle \tilde{f} \rangle)(\tilde{a}_1, \ldots, \tilde{a}_{2k}) = c_{m,p,p}^2 \Psi_{m,p}^k(\langle f \rangle)(a_1, \ldots, a_{2k}).$$

It is a crucial point here that the left-hand side be considered as a polynomial in the variables $x = (x_0, \ldots, x_m)$ and x_{m+1} that, by looking at the right-hand side, actually does not depend on the variable x_{m+1}. Taking the harmonic projection $H : \mathcal{P}_{m+2}^{2(p-k)} \to \mathcal{H}_{m+1}^{2(p-k)}$ of both sides (in all variables) we obtain that

$$H(\Psi_{m+1,p}^k(\langle \tilde{f} \rangle)(\tilde{a}_1, \ldots, \tilde{a}_{2k})) \neq 0$$

provided that $\Psi_{m,p}^k(\langle f \rangle)(a_1, \ldots, a_{2k}) \neq 0$. This latter relation is the consequence of Lemma 3.4.1 (with p replaced by $p-k+1$), in view of the fact that, for a polynomial $\xi \in \mathcal{P}^{2(p-k)}$, the harmonic part $H(\xi)$ is the top coefficient in the canonical decomposition of ξ.

Finally it remains for us to exhibit an example of a p-eigenmap $f : S^m \to S_V$, $m \ge 3$, that is isotropic of order $k-1$, $k \ge 1$, and $\Psi_{m,p}^k(\langle f \rangle) \neq 0$, since we then raise the domain dimension to obtain $\tilde{f}$ with the required nonvanishing property (3.5.23) (and land in domain dimensions ≥ 4). But the existence of this map follows from rigidity (Theorem 3.5.4). In fact, let $f : S^m \to S_V$ be a nonstandard p-eigenmap such that $\langle f \rangle \in V_{m+1}^{(2k,2k,0,\ldots,0)}$. By the induction hypothesis, f is isotropic of order $k-1$ and, by Corollary 3.5.8, $\Psi_{m,p}^k(\langle f \rangle)(a_1, \ldots, a_{2k}) \neq 0$ for some $a_1, \ldots, a_{2k} \in \mathbf{R}^{m+1}$. The second statement of Theorem 3.5.5 follows for $m \ge 4$.

The case $m = 3$, remains to be considered. Proceeding inductively, we assume that

$$\mathcal{F}_3^{p;k-1} = \sum_{(u,v) \in \triangle_k^p;\, u,v \text{ even}} V_4^{(u,v)}.$$

We want to show that $V_4^{(2(p-k),2k)}$ is disjoint from $\mathcal{F}_3^{p;k}$, or equivalently, $\Psi_p^k | V_4^{(2(p-k),2k)} \neq 0$. Let $\langle f \rangle \in V_4^{(2(p-k),2k)}$ with $f : S^3 \to S_V$ of boundary type. Assume, on the contrary, that $\Psi_p^k(\langle f \rangle) = 0$. By (3.5.31), $\Psi_p^k(\langle \tilde{f} \rangle) = 0$ so that, by what we proved above, $\tilde{f}$ is a minimal immersion of degree p and *order of isotropy* k: $\langle \tilde{f} \rangle \in \mathcal{F}_4^{p;k}$. For the domain-dimension-raising

operator Θ_3, we have $\Theta_3(\langle f\rangle) = \langle \tilde{f}\rangle$. On the other hand, Θ_3 must send $\langle f\rangle \in V_4^{(2(p-k),2k)}$ to zero since it is a homomorphism of $SO(4)$-modules and the multiplicity of $V_4^{(2(p-k),2k)}$ in $\mathcal{F}_4^{p;k}$ is zero, by branching (see Appendix 3). We obtain $\langle \tilde{f}\rangle = 0$; a contradiction. Theorem 3.5.5 follows.

The lower bound (3.3.8) for the range of quartic minimal immersions can be generalized to minimal immersions of highest possible isotropy as follows:

Theorem 3.5.10. *Let $p \geq 4$ even, and $f : S^m \to S_V$ a full minimal immersion of degree p and order of isotropy $p/2-1$. Then we have*

$$\dim V \geq \frac{\dim \mathcal{H}^{p-1}}{\dim \mathcal{H}^1} = \frac{(2p+m-3)(p+m-3)!}{(p-1)!(m-1)!(m+1)}. \tag{3.5.32}$$

PROOF. Applying (3.5.25) for $C = \langle f\rangle$ and $0 \leq k \leq p/2-1$, we see that $f^- : S^m \to S_{V\otimes\mathcal{H}^1}$ is a minimal immersion of degree $p-1$ and order of isotropy $p/2-1$. By rigidity (Theorem 3.5.4 for $k = p/2-1$), f^- is standard. Comparing ranges, we have

$$\dim(V \otimes \mathcal{H}^1) = \dim V \dim \mathcal{H}^1 \geq \dim \mathcal{H}^{p-1}.$$

Dividing by $\dim \mathcal{H}^1 = m+1$, we obtain (3.5.32).

Remark. Notice that, for $p = 4$, (3.5.32) reduces to (3.3.8).

3.6 Quartic Minimal Immersions in Domain Dimension Three

Theorem 3.1.6 asserts that the moduli space $\mathcal{M}_m^p$ (parametrizing the congruence classes of full homothetic minimal immersions $f : S^m \to S_V$ of degree p) is nontrivial iff $m \geq 3$ and $p \geq 4$. The main purpose of this section is to study the 18-dimensional moduli space $\mathcal{M}_3^4$, the lowest nonrigid range $m = 3$ and $p = 4$. Note that by Theorem 3.3.5, $\mathcal{M}_3^4$ recurs in $\mathcal{M}_3^p$ for all $p \geq 5$.

Within $\mathcal{M}_3^p$, $p \geq 4$, a prominent role will be played by the $SU(2)$-equivariant minimal immersions. The congruence classes of $SU(2)$-equivariant full minimal immersions $f : S^3 \to S_V$ of degree p are parametrized by the *"equivariant moduli space"* $(\mathcal{M}_3^p)^{SU(2)}$, the $SU(2)$-fixed points on $\mathcal{M}_3^p$ (Section 2.3). Since the linear span of $\mathcal{M}_3^p$ is $\mathcal{F}_3^p$, the equivariant moduli space is the linear slice

$$(\mathcal{M}_3^p)^{SU(2)} = \mathcal{M}_3^p \cap (\mathcal{F}_3^p)^{SU(2)}.$$

We can find $(\mathcal{F}_3^p)^{SU(2)}$ from the decomposition formula (3.1.11) for $\mathcal{F}_3^p$ by restricting both sides to $SU(2)$ and counting the trivial components. To do this, we need some preparation. Recall from Section 1.4 that W_p, $p \geq 0$, denotes the complex irreducible $SU(2)$-module with $W_p = p+$

1. Also, according to our conventions, for $v > 0$, $V^{(u,v)}$ stands for the sum $V_4^{(u,v)} \oplus V_4^{(u,-v)}$ of $SO(4)$-modules. Our immediate aim is to describe the components of this splitting in terms of representations of $SU(2)$ and $SU(2)'$. In what follows we use the notations introduced in Section 2.7.

Proposition 3.6.1. *Let $u \geq v > 0$ and $u + v$ even. We have*

$$V^{(u,v)} = (W_{u-v} \otimes W'_{u+v}) \oplus (W_{u+v} \otimes W'_{u-v})$$

as $SO(4)$-modules. In particular

$$V^{(u,v)}|_{SU(2)} = (u + v + 1)W_{u-v} \oplus (u - v + 1)W_{u+v}.$$

PROOF. The northern vertex (u, v) of the triangle $\triangle_0^{u,v}$ in (3.1.7) is missed by the subtriangles $\triangle_0^{u-1,v-1}$ and $\triangle_0^{u+1,v-1}$ overlapping in $\triangle_0^{u,v-2}$. Looking at (3.1.7) again, we obtain

$$V^{(u,v)} \oplus (\mathcal{H}^{u-1} \otimes \mathcal{H}^{v-1}) \oplus (\mathcal{H}^{u+1} \otimes \mathcal{H}^{v-1}) = (\mathcal{H}^u \otimes \mathcal{H}^v) \oplus (\mathcal{H}^u \otimes \mathcal{H}^{v-2}).$$

Each tensor product can be worked out using the Clebsch-Gordan formula (see Fulton-Harris [1], Vilenkin [1], or Section 2.7):

$$\begin{aligned} \mathcal{H}^r \otimes \mathcal{H}^s &= (W_r \otimes W'_r) \otimes (W_s \otimes W'_s) \\ &= (W_{r+s} \oplus W_{r+s-2} \oplus \ldots \oplus W_{r-s}) \\ &\quad \otimes (W'_{r+s} \oplus W'_{r+s-2} \oplus \ldots \oplus W'_{r-s}), \ r \geq s. \end{aligned}$$

Putting these together, the proposition follows.

Corollary 3.6.2. *Let $u \geq v \geq 1$ and $u + v$ be even. Then $V^{(u,v)}|_{SU(2)}$ contains the trivial $SU(2)$-module iff $u = v$. The multiplicity of the trivial $SU(2)$-module in $V^{(u,u)}$ is $2u + 1$. More generally, as $SO(4)$-modules, we have*

$$V^{(u,u)} = W_{2u} \oplus W'_{2u}.$$

By this corollary, the decomposition in (3.1.11) gives

$$\begin{aligned} (\mathcal{F}_3^p \otimes \mathbf{C})^{SU(2)} &= \sum_{l=2}^{[p/2]} (V^{(2l,2l)})^{SU(2)} \\ &= \sum_{l=2}^{[p/2]} (W_{4l} \oplus W'_{4l})^{SU(2)} \\ &= \sum_{l=2}^{[p/2]} W'_{4l} \end{aligned}$$

as $SU(2)'$-modules. As real $SU(2)'$-modules, we thus have

$$(\mathcal{F}_3^p)^{SU(2)} = \sum_{l=2}^{[p/2]} R'_{4l}.$$

Counting dimensions, we obtain

$$\begin{aligned}\dim(\mathcal{M}_3^p)^{SU(2)} = \dim(\mathcal{F}_3^p)^{SU(2)} &= \sum_{l=2}^{[p/2]} R'_{4l} \\ &= \sum_{l=2}^{[p/2]} (4l+1) \\ &= \left(2\left[\frac{p}{2}\right]+5\right)\left(\left[\frac{p}{2}\right]-1\right).\end{aligned}$$

Remark. This formula was first derived by DeTurck-Ziller [1] using a "partially heuristic argument". The exact computation above is due to Toth-Ziller [1].

In a similar vein, we have

$$(\mathcal{F}_3^p \otimes \mathbf{C})^{SU(2)'} = \sum_{l=2}^{[p/2]} W_{4l},$$

and as real $SU(2)$-modules:

$$(\mathcal{F}_3^p)^{SU(2)'} = \sum_{l=2}^{[p/2]} R_{4l}.$$

Remark. It is also clear that, for $SU(2)$- (and $SU(2)'$-)equivariant p-eigenmaps, we have

$$(\mathcal{E}_3^p)^{SU(2)} = \sum_{l=1}^{[p/2]} R'_{4l} \quad \text{and} \quad (\mathcal{E}_3^p)^{SU(2)'} = \sum_{l=1}^{[p/2]} R_{4l}.$$

As a byproduct, for $p = 2$, we obtain (2.7.9).

Summarizing, the $SU(2)$-equivariant spherical minimal immersions of degree p are parametrized by the $SU(2)'$-invariant slice

$$(\mathcal{M}_3^p)^{SU(2)} = \sum_{l=2}^{[p/2]} R'_{4l} \cap \mathcal{M}_3^p$$

and the $SU(2)'$-equivariant ones by the $SU(2)$-invariant slice

$$(\mathcal{M}_3^p)^{SU(2)'} = \sum_{l=2}^{[p/2]} R_{4l} \cap \mathcal{M}_3^p.$$

The diagonal element $\gamma = \operatorname{diag}(1,1,1,-1) \in O(4)$ (conjugating $SU(2)$ to $SU(2)'$ (cf. Section 2.7)) interchanges $(\mathcal{M}_3^p)^{SU(2)}$ and $(\mathcal{M}_3^p)^{SU(2)'}$. The sum $\sum_{k=2}^{[p/2]} R_{4k} \oplus R'_{4k}$ complexified is parametrized by the northwestern edge of

$\triangle_2^p$ in (3.1.11). In the following discussion we will restrict ourselves to the "equivariant slice"

$$\mathcal{N}_3^p = \sum_{l=2}^{[p/2]} (R_{4l} \oplus R'_{4l}) \cap \mathcal{M}_3^p.$$

Clearly, $(\mathcal{N}_3^p)^{SU(2)} = (\mathcal{M}_3^p)^{SU(2)}$ and $(\mathcal{N}_3^p)^{SU(2)'} = (\mathcal{M}_3^p)^{SU(2)'}$, and $\mathcal{N}_3^4 = \mathcal{M}_3^4$ (since $\triangle_2^4$ reduces to a point). Taking boundaries in various spaces, we obtain

$$\partial(\mathcal{N}_3^p)^{SU(2)} = \sum_{l=2}^{[p/2]} R'_{4l} \cap \partial\mathcal{M}_3^p$$

and

$$\partial(\mathcal{N}_3^p)^{SU(2)'} = \sum_{l=2}^{[p/2]} R_{4l} \cap \partial\mathcal{M}_3^p.$$

Proposition 3.6.3. *$\partial\mathcal{N}_3^p$ is the union of line segments with one endpoint on $\partial(\mathcal{N}_3^p)^{SU(2)}$ and the other on $\partial(\mathcal{N}_3^p)^{SU(2)'}$. Equivalently, every full minimal immersion $f : S^3 \to S_V$ of degree p such that $\langle f \rangle \in \mathcal{N}_3^p$ is congruent to $(\sqrt{\lambda_1} f_1, \sqrt{\lambda_2} f_2) : S^3 \to S_{V_1 \times V_2}$, $\lambda_1 + \lambda_2 = 1$, $\lambda_1, \lambda_2 \geq 0$, where $f_1 : S^3 \to S_{V_1}$ and $f_2 : S^3 \to S_{V_2}$ are full minimal immersions of degree p, with f_1 $SU(2)$-equivariant and f_2 $SU(2)'$-equivariant.*

PROOF. The equivalence of the two statements follows from Theorem 2.3.5. For the first statement, observe that it is enough to prove that any line segment connecting $\partial(\mathcal{N}_3^p)^{SU(2)}$ and $\partial(\mathcal{N}_3^p)^{SU(2)'}$ is contained entirely in the boundary of $\mathcal{M}_3^p$. Let the endpoints be $\langle f_1 \rangle$ and $\langle f_2 \rangle$ as in the second statement, and let $\langle f \rangle$ be a point in the interior of the line segment. We need to show that $\langle f \rangle \in \partial\mathcal{M}_3^p$, or equivalently, that V_f is a proper linear subspace of $\mathcal{H}_3^p$. By Theorem 2.3.5 cited above, we have

$$V_f = V_{f_1} + V_{f_2}.$$

By assumption, $V_{f_1} \subset \mathcal{H}_3^p|_{SU(2)}$ and $V_{f_2} \subset \mathcal{H}_3^p|_{SU(2)'}$ are proper submodules. Complexifying and using Lemma 2.7.9, we see that V_f is a proper linear subspace of $\mathcal{H}_3^p$. The proposition follows.

We obtain the following generalization of the first statement of Theorem 2.7.5:

Corollary 3.6.4. *$\mathcal{N}_3^p$ is the convex hull of $(\mathcal{N}_3^p)^{SU(2)}$ and $(\mathcal{N}_3^p)^{SU(2)'}$. In particular, $\mathcal{M}_3^4$ is the convex hull of $(\mathcal{M}_3^4)^{SU(2)}$ and $(\mathcal{M}_3^4)^{SU(2)'}$.*

Remark 1. As in the case of quadratic eigenmaps, we see that, for a full quartic $SU(2)$-equivariant minimal immersion $f : S^3 \to S_V$, we have

$$(\mathcal{M}_f)^{SU(2)} = \mathcal{M}_f.$$

Remark 2. Let $f : S^3 \to S_V$ be a full $SU(2)$-equivariant minimal immersion of degree p even and order of isotropy $p/2 - 1$ (Theorem 3.5.5). In contrast to Remark 1 above, we claim that, for $p \geq 130$, we have

$$\dim(\mathcal{M}_f)^{SU(2)} + 63,828 \leq \dim \mathcal{M}_f.$$

(This should also be compared to the result of Escher-Weingart [1] which provides an example of a full $SU(2)$-equivariant minimal immersion $f : S^3 \to S_V$ of degree 36 such that $(\mathcal{M}_f)^{SU(2)}$ is trivial but $\dim \mathcal{M}_f \geq 9$ (cf. Remark 2 in Section 1.5).)

For the proof, we first note that, by (3.1.4), we have

$$\dim S^2(V_f) \leq \dim S^2(\mathcal{H}^p)/\mathcal{F}^p + \dim \mathcal{F}_f.$$

Since $\dim S^2(V_f) = \dim S^2 V$ (f is full) and $\dim \mathcal{F}_f = \dim \mathcal{M}_f$, we obtain the lower estimate

$$\dim S^2 V - \dim S^2(\mathcal{H}^p)/\mathcal{F}^p \leq \dim \mathcal{M}_f.$$

Using (3.1.8)-(3.1.9), Theorem 3.1.6, and the Weyl dimension formula (Appendix 3), we compute

$$\begin{aligned}
\dim S^2(\mathcal{H}^p)/\mathcal{F}^p &= \dim \mathcal{P}^{2p} + \sum_{j=1}^{p-1} \dim_{\mathbf{C}} V^{(2j,2)} \\
&= \binom{2p+3}{3} + \sum_{j=1}^{p-1} (2j-1)(2j+3) \\
&= \binom{2p+3}{3} + \frac{2(p-1)(4p^2+4p-9)}{3}.
\end{aligned}$$

On the other hand, substituting $m = 3$ in (3.5.32), we have

$$\dim V \geq \frac{p^2}{4}.$$

Putting all these together, our lower estimate for the moduli space becomes

$$\frac{p^2(p^2+4)}{32} - \binom{2p+3}{3} - \frac{2(p-1)(4p^2+4p-9)}{3} \leq \dim \mathcal{M}_f.$$

Finally, since p is even, our computation after Corollary 3.6.2 gives

$$\dim(\mathcal{M}_f)^{SU(2)} \leq \dim(\mathcal{M}^p)^{SU(2)} = \frac{(p+5)(p-2)}{2}.$$

Combining the last two estimates, we obtain

$$\begin{aligned}
&\frac{p^2(p^2+4)}{32} - \binom{2p+3}{3} \\
&\quad - \frac{2(p-1)(4p^2+4p-9)}{3} - \frac{(p+5)(p-2)}{2} \\
&\qquad \leq \dim(\mathcal{M}_f)^{SU(2)} - \dim \mathcal{M}_f.
\end{aligned}$$

The quartic polynomial on the left-hand side is strictly increasing for $p \geq 97$, has largest real root 129.077..., and its value at $p = 130$ is $63,828$.[1] The claim now follows.

We now return to the main line. Since $\partial(\mathcal{N}_3^p)^{SU(2)'}$ is a copy of $\partial(\mathcal{N}_3^p)^{SU(2)}$, to describe $\partial\mathcal{N}_3^p$ we may restrict our study to $SU(2)$-equivariant minimal immersions. From now on, unless stated otherwise, all minimal immersions $f : S^3 \to S_V$ will be $SU(2)$-equivariant. Then V is an $SU(2)$-submodule of $\mathcal{H}^p|_{SU(2)}$. On the other hand, $\mathcal{H}^p|_{SU(2)} = W_p \otimes W_p'|_{SU(2)} = (p+1)W_p$ so that V (complexified) must be a multiple of W_p. For p even, $\mathcal{H}^p|_{SU(2)} = R_p \otimes R_p'|_{SU(2)} = (p+1)R_p$ as real $SU(2)$-modules, so that V is a multiple of R_p.

In Section 1.4, we introduced the equivariant construction in our study of $SU(2)$-equivariant eigenmaps. Recall that given a polynomial ξ of unit length in $\mathcal{H}_3^p$, the $SU(2)$-equivariant eigenmap f_ξ is the orbit map of ξ given by (1.4.2). The immersion f_ξ is minimal iff it is conformal and, by $SU(2)$-equivariance, this holds iff f_ξ is homothetic at the identity.

The systems of equations (1.4.13) and (1.4.16) for the coefficients of the polynomial ξ in (1.4.1) give the necessary and sufficient conditions for f_ξ to be homothetic at the identity. In the following discussion, we study the solvability of these systems in specific instances.

As a first application, we see that $V = R_4$ cannot occur for a full quartic minimal immersion $f : S^3 \to S_V$. Indeed, for $p = 2d = 4$, (1.4.16) reduces to the following

$$\begin{aligned} 48|c_0|^2 + 12|c_1|^2 + 4r^2 &= 1, \\ 96{c_0}^2 + 6|c_1|^2 &= 1, \\ 3c_1^2 + 8c_0 r &= 0 \\ 6c_0\bar{c}_1 - c_1 r &= 0. \end{aligned}$$

Simple inspection shows that these equations are inconsistent.

Remark. Nonexistence of $SU(2)$-equivariant quartic minimal immersions $f : S^3 \to S_{R_4}$ also follow from Moore's result that states that a minimal immersion $f : S^m \to S^n$ with $n \leq 2m - 1$ is totally geodesic (Section 1.5). (For yet another proof, see Problem 3.16.)

In analogy with Corollary 2.7.10, we have the following

Proposition 3.6.5. *The possible spherical range dimensions of a (not necessarily equivariant) full quartic minimal immersion $f : S^3 \to S_V$ are*

$$\dim S_V = 9, 14, 15, 18 - 24.$$

f is $SU(2)$- or $SU(2)'$-equivariant iff $\dim V = \dim S_V + 1$ is divisible by 5.

[1] Here the use of a computer algebra system is recommended.

PROOF. If f is $SU(2)$-equivariant, then V is a multiple of R_4. Since $V = R_4$ is not realized, the possible dimensions of V are 10, 15, 20, 25. Without the $SU(2)$-equivariance, according to Proposition 3.6.3, the space of components V_f can be written as

$$V_f = V_{f_1} + V_{f_2},$$

where $f_1 : S^3 \to S_{V_1}$ and $f_2 : S^3 \to S_{V_2}$ are full quartic minimal immersions with f_1 $SU(2)$-equivariant, and f_2 $SU(2)'$-equivariant. Since V_{f_1} is an $SU(2)$-submodule of $\mathcal{H}^4 = R_4 \otimes R_4'$, we are in a position to apply Lemma 2.7.9 and obtain $V_{f_1} = R_2 \otimes W_0'$, where $W_0' \subset R_4'$ is a linear subspace of dimension ≥ 2. Similarly, $V_{f_2} = W_0 \otimes R_2'$, where $W_0 \subset R_4$ is a linear subspace of dimension ≥ 2. We now compute

$$\begin{aligned}\dim V &= \dim V_f \\ &= \dim R_4 \otimes W_0' + \dim W_0 \otimes R_4' \\ &\quad - \dim W_0 \otimes W_0' \\ &= 5(\dim W_0 + \dim W_0') - \dim W_0 \dim W_0'.\end{aligned}$$

Since $2 \leq \dim W_0, \dim W_0' \leq 5$, the proposition follows.

$SU(2)$-equivariant quartic minimal immersions do exist for $V = W_4 = 2R_4$. In particular, a case-by-case check shows that all range dimensions in Proposition 3.6.5 are realized. For $p = 4$, (1.4.13) reduces to

$$\begin{aligned}&48|c_0|^2 + 3|c_1|^2 + 3|c_3|^2 + 48|c_4|^2 = 1, \\ &24|c_0|^2 + 6|c_1|^2 + 4|c_2|^2 + 6|c_3|^2 + 24|c_4|^2 = 1, \\ &4c_0\bar{c}_2 + 3c_1\bar{c}_3 + 4c_2\bar{c}_4 = 0, \\ &6c_0\bar{c}_1 + c_1\bar{c}_2 - c_2\bar{c}_3 - 6c_3\bar{c}_4 = 0.\end{aligned} \tag{3.6.1}$$

As a specific example, let

$$c_0 = \frac{\sqrt{6}}{24},\ c_1 = 0,\ c_2 = \frac{\sqrt{2}}{4},\ c_3 = 0,\ c_4 = -\frac{\sqrt{6}}{24}.$$

The corresponding polynomial is

$$\xi = \frac{\sqrt{6}}{24}(z^4 - w^4) + \frac{\sqrt{2}}{4}z^2w^2.$$

To work out the orbit map $f_\xi : S^3 \to S_{W_4}$, we identify W_4 with $\mathbf{C}^5$ by the orthonormal basis

$$\frac{1}{\sqrt{24}}z^4,\ \frac{1}{\sqrt{6}}z^3w,\ \frac{1}{2}z^2w^2,\ \frac{1}{\sqrt{6}}zw^3,\ \frac{1}{\sqrt{24}}w^4.$$

We obtain the full quartic minimal immersion[2] $\mathcal{I} : S^3 \to S^9$ given by

$$\begin{aligned}\mathcal{I}(z,w) = \Big(&\frac{1}{\sqrt{2}}(z^4 - \bar{w}^4), \sqrt{6}z^2\bar{w}^2,\\ &\sqrt{2}(z^3w + \bar{z}\bar{w}^3), \sqrt{6}(z\bar{z}^2w - \bar{z}w^2\bar{w}),\\ &\sqrt{\frac{3}{2}}(z^2w^2 - \bar{z}^2\bar{w}^2), \frac{1}{\sqrt{2}}(|z|^4 - 4|z|^2|w|^2 + |w|^4)\Big).\end{aligned} \tag{3.6.2}$$

The first four coordinates are complex, the fifth is purely imaginary, and the sixth is real so that $\mathcal{I}$ maps into $\mathbf{C}^4 \times (i\mathbf{R}) \times \mathbf{R} = \mathbf{R}^{10}$. The minimal immersion $\mathcal{I} : S^3 \to S^9$ will play a prominent role (analogous to that of the Hopf map for quadratic eigenmaps; cf. Example 1.4.2) in understanding the structure of the boundary $\partial(\mathcal{M}_3^4)^{SU(2)}$.

Remark. The invariance group of ξ, the subgroup of $SU(2)$ that leaves ξ invariant, is (conjugate to) the quaternionic group $\mathbf{D}_2^* = \{\pm 1, \pm i, \pm j \pm k\}$. Thus, factoring out, we obtain a minimal imbedding $f_\xi : S^3/\mathbf{D}_2^* \to S_{W_4}$ of the prism manifold $S^3/\mathbf{D}_2^*$ into S^9. This is not the minimal codimension example for $S^3/\mathbf{D}_2^*$ (realized by a degree 8 minimal immersion into S^8, cf. the table at the end of Section 1.5).

The following analogue of Corollary 2.7.2 is due to DeTurck-Ziller [1] (based on an idea of Mashimo [1]).

Theorem 3.6.6. *Given a full $SU(2)$-equivariant quartic minimal immersion $f : S^3 \to S^9$, we have $f = U \circ \mathcal{I} \circ g$ for some $U \in O(10)$ and $g \in SU(2)'$.*

PROOF. We prove that among the $SU(2)$-orbits of polynomials in W_4, up to isometry, there is a unique orbit of constant curvature $3/\lambda_4 = 1/8$ (Proposition 3.1.1). To do this we first observe that, given a polynomial $\xi_0 \in W_4$, with normal space ν of the orbit $SU(2)(\xi_0)$ at ξ_0, every other $SU(2)$-orbit must intersect ν. This is because all orbits are at a constant distance from each other so that the minimal geodesic in W_4 connecting two of these is orthogonal to the orbits. To be specific, let $\xi_0 = z^4$. To determine the tangent space $T_{\xi_0}(SU(2)(\xi_0))$, we use the basis (1.4.12) of $su(2)$. Since the 1-parameter subgroup in $S^3 = SU(2)$ corresponding to $Z \in su(2)$ is $t \mapsto e^{it}$, $t \in \mathbf{R}$, we obtain

$$Z_{\xi_0} = \frac{d}{dt}(e^{it}z)^4|_{t=0} = 4iz^4.$$

Similarly, we have

$$X_{\xi_0} = \frac{d}{dt}(\cos(t)z + \sin(t)w)^4|_{t=0} = 4z^3w$$

[2]This should not be confused with the icosahedral form $\mathcal{I}$ defined in Section 1.4.

and

$$Y_{\xi_0} = 4iz^3w.$$

We conclude that $T_{\xi_0}(SU(2)(\xi_0))$ is spanned by $\{iz^4, z^3w, iz^3w\}$. We now let $\xi \in \nu$. As usual, we set

$$\xi(z,w) = c_0z^4 + c_1z^3w + c_2z^2w^2 + c_3zw^3 + c_4w^4.$$

Since $\xi \perp T_{\xi_0}(SU(2)(\xi_0))$, we have

$$c_1 = 0.$$

With this, the last two equations in (3.6.1) reduce to

$$c_0\bar{c}_2 = -c_2\bar{c}_4, \text{ and } c_2\bar{c}_3 = -6c_3\bar{c}_4. \tag{3.6.3}$$

We claim that $c_2c_3 = 0$. Indeed, if both c_2 and c_3 were nonzero then (3.6.3) would imply $|c_0|^2 = |c_4|^2$ and $|c_2|^2 = 36|c_4|^2$. These are inconsistent with the first two equations in (3.6.1). In a similar vein, c_2 and c_3 cannot vanish simultaneously. Notice finally that acting on the polynomial ξ by $\operatorname{diag}(e^{i\theta}, e^{-i\theta})$ and by the isometry $c_q \mapsto e^{i\phi}c_q$, $q = 0, \ldots, 4$, we can leave the condition $c_1 = 0$ invariant and provide two degrees of freedom to make any two of the remaining variables c_0, c_2, c_3, c_4 either (positive) real or purely imaginary (with positive imaginary part). If $c_2 = 0$ and $c_3 \neq 0$ then $c_4 = 0$ and, assuming that c_0 and c_3 are real, inspection of (3.6.1) gives us

$$\xi_1 = \frac{\sqrt{2}}{12}z^4 + \frac{1}{3}zw^3.$$

If $c_3 = 0$ and $c_2 \neq 0$ then we assume that $c_4 \in \mathbf{R}$ and $c_2 \in i\mathbf{R}$. Then $c_0 = c_4$ and again inspection of (3.6.1) gives us

$$\xi_2 = \frac{\sqrt{6}}{24}(z^4 + w^4) + \frac{i}{8}z^2w^2.$$

Summarizing, we obtain that, up to congruence, the only $SU(2)$-orbits of polynomials in W_4 of constant curvature $1/8$ are $SU(2)(\xi_1)$ and $SU(2)(\xi_2)$.

Finally, we claim that these two orbits are congruent. Indeed, we can transform ξ_1 to a polynomial of the form $c_0z^4 + c_2z^2w^2 + c_4w^4$ by a suitable element in $SU(2)$. The argument for the case $c_3 = 0$ now applies and gives that $SU(2)(\xi_1)$ can be carried into $SU(2)(\xi_2)$ by a suitable isometry on W_4. The theorem follows.

Remark. In Theorem 3.6.6 $SU(2)$-equivariance can be dispensed with. This is a simple consequence of the proof of Proposition 3.6.5 since a quartic minimal immersion $f : S^3 \to S^9$ is necessarily $SU(2)$- or $SU(2)'$-equivariant.

Since 9 is the least range dimension *among the ($SU(2)$-equivariant) quartic minimal immersions*, those with range dimension 9 are linearly rigid.

Since the moduli space is the convex hull of points corresponding to linearly rigid minimal immersions (Theorem 3.1.4), it is natural to ask whether there exist full $SU(2)$-equivariant linearly rigid quartic minimal immersions with range dimension > 9. We will see later that the answer is affirmative. This is in contrast to the case of quadratic eigenmaps, where linear rigidity is present only in the minimum range dimension.

The following proposition implies that, in the quartic case, linear rigidity may only exist in range dimensions 9 and 14.

Proposition 3.6.7. *Assume that $p \geq 4$ is even. Let $f : S^3 \to S_V$ be a full $SU(2)$-equivariant minimal immersion of degree p, and write $V = kR_p$, $k = 1, \ldots, p+1$. We have*

$$\dim(\mathcal{M}_f)^{SU(2)} \geq \frac{k(k+1)}{2} - 6.$$

In particular, f is linearly nonrigid if $k \geq 4$.

PROOF. We consider the Lie algebra $su(2)$ as the tangent space of S^3 at the identity. For $U \in su(2)$, we denote by U_R the right invariant extension of U on S^3. Given $C \in S^2V$, we define the linear map

$$\Psi(C) : su(2) \times su(2) \to \mathcal{P}^{2p},$$

by

$$\Psi(C)(U, U') = \langle Cf_*(U_R)\check{}, f_*(U'_R)\check{}\rangle = \langle CU_R(f), U'_R(f)\rangle.$$

Evaluating $U_R(f)$ for $U = Z, X, Y$, the elements of the standard basis of $su(2)$ in (1.4.12), it follows easily that this is a homogeneous polynomial of degree $2p$, i.e. it belongs to $\mathcal{P}^{2p}$. For example:

$$Z_{R,z+jw} = -x_1\partial_0 + x_0\partial_1 + x_3\partial_2 - x_2\partial_3,$$

where $z = x_0 + ix_1$ and $w = x_2 + ix_3$. Thus, $\Psi(C)$ maps into $\mathcal{P}^{2p}$. Since $\Psi(C)$ is symmetric in the arguments U and U', it can be considered as a linear map

$$\Psi(C) : S^2(su(2)) \to \mathcal{P}^{2p},$$

or equivalently, an element $\Psi(C) \in \mathcal{P}^{2p} \otimes S^2(su(2)^*)$. We now vary C in S^2V and obtain the linear map

$$\Psi : S^2V \to \mathcal{P}^{2p} \otimes S^2(su(2)^*).$$

Since the right invariant vector fields (pointwise) span each tangent space in S^3, we have

$$\ker \Psi = \mathcal{F}_f.$$

To estimate this kernel we first claim that Ψ is a homomorphism of $SU(2)$-modules, where the module structure on V is given by the

$SU(2)$-equivariance of f. Explicitly, for $g \in SU(2)$, we have

$$\Psi(g \cdot C)(Ad(g)(U), Ad(g)(U')) = \Psi(C)(U, U') \circ L_{g^{-1}}.$$

To show this we let R_g denote right quaternionic multiplication with $g \in S^3$. Using $Ad(g) = (L_g)_* \circ (R_{g^{-1}})_*$, we calculate, at $x \in S^3$:

$$\begin{aligned}
(Ad(g)\,U)_{R,x}(f \circ L_{g^{-1}}) &= ((R_x)_*(Ad(g)U_1))(f \circ L_{g^{-1}}) \\
&= ((L_g)_* \circ (R_{g^{-1}})_*)(U_1)(f \circ L_{g^{-1}} \circ R_x) \\
&= U_1(f \circ R_x \circ R_{g^{-1}}) \\
&= (R_{g^{-1}x})_*(U_1)(f) \\
&= U_{R,g^{-1}x}(f) \\
&= ((U_R(f)) \circ L_{g^{-1}})(x).
\end{aligned}$$

The claim follows.

By assumption, we have $V = kR_p$ as $SU(2)$-modules. Thus

$$S^2V = S^2(kR_p) = kS^2(R_p) \oplus \frac{k(k-1)}{2}(R_p \otimes R_p).$$

We now count the trivial $SU(2)$-components. Since the trivial $SU(2)$-module R_0 is contained in both $S^2(R_p)$ and $R_p \otimes R_p$, the multiplicity of R_0 in S^2V is at least $k + k(k-1)/2 = k(k+1)/2$. On the other hand,

$$\begin{aligned}
\mathcal{P}^{2p} &= \mathcal{H}^{2p} \oplus \mathcal{H}^{2p-2} \oplus \ldots \oplus \mathcal{H}^2 \oplus \mathcal{H}^0 \\
&= (2p+1)R_{2p} \oplus (2p-1)R_{2p-2} \oplus \ldots \oplus 3R_2 \oplus R_0,
\end{aligned}$$

where the first equality is an isomorphism of $SO(4)$-modules, the second is an isomorphism of $SU(2)$-modules. Finally, $su(2)^* = R_2$ so that

$$S^2(su(2)^*) = R_4 \oplus R_0.$$

Putting these together, we have

$$\mathcal{P}^{2p} \otimes S^2(su(2)^*) = (\mathcal{P}^{2p} \otimes R_4) \oplus \mathcal{P}^{2p}.$$

By the Clebsch-Gordan formula, $R_r \otimes R_s$, $r \geq s \geq 0$, contains R_0 iff $r = s$. Thus the multiplicity of R_0 in $\mathcal{P}^{2p} \otimes S^2(su(2)^*)$ is $(4+1)+1 = 6$. Comparing this with the domain of Ψ, we see that $((k(k+1)/2) - 6)R_0$ must be in the kernel.

Remark. Proposition 3.6.7 should be compared to Theorem 2.7.8, in fact, we could have proved it using (3.1.4) (Problem 3.15). Note further that the lower estimate in Proposition 3.6.7 is sharp. In fact, for the standard minimal immersion, we have $k = p + 1$, and the lower bound works out to be $(p+5)(p/2-1)$. By the computations after Corollary 3.6.2, for p being even, this is the dimension of $(\mathcal{M}_3^p)^{SU(2)}$.

Let $f : S^3 \to S_V$ be a full $SU(2)$-equivariant quartic minimal immersion. Assume that f is linearly rigid. By Proposition 3.6.7, we have $V = kR_4$

with $k = 2, 3$. For $k = 2$, $\langle f \rangle$ is in the $SU(2)$-orbit of $\langle \mathcal{I} \rangle$ (Theorem 3.6.6). As for $k = 3$, we now exhibit an example of a full linearly rigid $SU(2)$-equivariant quartic minimal immersion $f : S^3 \to S_V$ with $V = 3R_4$. In fact, we claim that the antipodal $\mathcal{I}^o$ of $\mathcal{I}$ (Section 2.3) is linearly rigid and has range dimension 14.

To prove this claim, first notice that $\mathcal{I}$ has orthonormal components. This follows by evaluating the scalar product of each pair of components in (3.6.2) using (2.1.27). Theorem 2.3.19 then implies that $\mathcal{I}^o$ has range dimension 14. We show that $\mathcal{I}^o$ is linearly rigid by contradiction. Assume that $\mathcal{M}_{\mathcal{I}^o} = (\mathcal{M}_{\mathcal{I}^o})^{SU(2)}$ is nontrivial and consider a line segment through $\langle \mathcal{I}^o \rangle$ with endpoints $\langle f_1 \rangle$ and $\langle f_2 \rangle$ on $\partial \mathcal{M}_{\mathcal{I}^o}$. f_1 and f_2 must have range dimension 9. Consequently, the antipodals f_1^o and f_2^o have range dimension 14 (Theorem 3.6.6). Let $\langle f \rangle$ be the intersection of the segment connecting $\langle f_1^o \rangle$ and $\langle f_2^o \rangle$ with the line $\mathbf{R} \cdot \langle \mathcal{I}^o \rangle$. We claim that f has range dimension 19, and this gives a contradiction since it should be congruent to $\mathcal{I}$ (the antipodal of $\mathcal{I}^o$) having range dimension 9. To prove the claim, we use Theorem 2.3.19 again, and compute

$$\begin{aligned} V_f &= V_{f_1^o} + V_{f_2^o} \\ &= V_{f_1}^{\perp} + V_{f_2}^{\perp} \\ &= (V_{f_1} \cap V_{f_2})^{\perp}. \end{aligned}$$

On the other hand

$$\begin{aligned} \dim(V_{f_1} \cap V_{f_2}) &= \dim V_{f_1} + \dim V_{f_2} \\ &\quad - \dim(V_{f_1} + V_{f_2}) \\ &= 10 + 10 - \dim V_{\mathcal{I}^o} \\ &= 10 + 10 - (25 - 10) = 5. \end{aligned}$$

The claim and linear rigidity of $\mathcal{I}^o$ follows.

We see that the convex hull of the $SU(2)'$-orbit of $\langle \mathcal{I} \rangle$ and its (orthogonal) γ-image is *properly contained* in $\mathcal{M}_3^4$ since there are linearly rigid full quartic minimal immersions with nonminimal range dimension (Theorem 3.1.4).

We say that a full $SU(2)$-equivariant quartic minimal immersion $f : S^3 \to S_V$ is of type I, II, or III if $\dim V = 10$, 15 or 20. We denote by **I**, **II** and **III** the subsets of $(\mathcal{M}_3^4)^{SU(2)}$ that correspond to all quartic minimal immersions of type I, II, and III.

Theorem 3.6.6 can be reformulated by saying that **I** is a single $SU(2)'$-orbit through $\langle \mathcal{I} \rangle$. As for the topological structure of **I**, we have the following:

Theorem 3.6.8. *As a homogeneous space,* **I** *is an octahedral manifold* $S^3/\mathbf{O}^*$, *where* $\mathbf{O}^*$ *is the binary octahedral group. As an* $SU(2)'$*-orbit in* $(\mathcal{M}_3^4)^{SU(2)} \cong R_8'$, **I** *is imbedded minimally in an* 8*-sphere of* R_8'. *The imbedding is given by an* $SU(2)$*-equivariant minimal immersion of degree* 8.

To prove Theorem 3.6.8 we first have to show that $\mathbf{O}^*$ is the isotropy subgroup of the $SU(2)'$ action at $\langle \mathcal{I} \rangle$. This will follow as a byproduct of a more general computation to be carried out later in this section. Once this is done, it will follow that the orbit $S^3/\mathbf{O}^*$ must be minimally imbedded in a sphere. In fact, as was observed in Section 1.5 (Remark 1), $S^3/\mathbf{O}^*$ is isotropy irreducible and hence, for every invariant polynomial, the equivariant construction must give an isometric imbedding.

The set $\mathbf{II}$ splits into the disjoint union

$$\mathbf{II} = \mathbf{II}_0 \cup \mathbf{II}_1 \tag{3.6.4}$$

corresponding to linearly rigid and nonrigid quartic minimal immersions. By the above, $\dim \mathbf{II}_0 \geq 3$ since the $SU(2)'$-orbit of $\langle \mathcal{I}^o \rangle$ is contained in $\mathbf{II}_0$.

Theorem 3.6.9. *We have*

$$\dim \mathbf{II} \leq 6. \tag{3.6.5}$$

Moreover, we have

$$\dim \mathbf{II}_0 = 6. \tag{3.6.6}$$

and

$$\mathbf{II}_1 = SU(2)' \cdot \mathcal{D}, \tag{3.6.7}$$

where $\mathcal{D}$ is a flat 2-dimensional disk with boundary circle on the octahedral manifold $\mathbf{I}$.

PROOF. We show (3.6.5) by a careful dimension computation. For a type II minimal immersion $f : S^3 \to S_V$ the range $SU(2)$-module is $3R_4 (= V)$. Since R_4 can be thought of as the $SU(2)$-module of quartic polynomials

$$\xi(z, w) = a_0 z^4 + \bar{a}_0 w^4 + a_1 z^3 w - \bar{a}_1 z w^3 - r z^2 w^2, \ a_0, a_1 \in \mathbf{C}, \ r \in \mathbf{R},$$

(Section 1.4), we look upon a general element of $3R_4$ as a triple:

$$\begin{pmatrix} a_0 z^4 + \bar{a}_0 w^4 + a_1 z^3 w - \bar{a}_1 z w^3 - r z^2 w^2 \\ b_0 z^4 + \bar{b}_0 w^4 + b_1 z^3 w - \bar{b}_1 z w^3 - s z^2 w^2 \\ c_0 z^4 + \bar{c}_0 w^4 + c_1 z^3 w - \bar{c}_1 z w^3 - t z^2 w^2 \end{pmatrix}$$

where $a_0, a_1, b_0, b_1, c_0, c_1 \in \mathbf{C}$ and $r, s, t \in \mathbf{R}$. The decomposition of V as $3R_4$ is not unique, since $SO(3)$ acts on $3R_4$ in a natural way. Rotating $(r, s, t) \in \mathbf{R}^3$, we may assume that $r = s = 0$ and $t \geq 0$. We still have the freedom to rotate about the third axis. This amounts to the change

$$\begin{aligned} a_0 &\mapsto \cos\alpha \cdot a_0 - \sin\alpha \cdot b_0, \\ b_0 &\mapsto \sin\alpha \cdot a_0 + \cos\alpha \cdot b_0, \end{aligned} \tag{3.6.8}$$

and similarly for a_1 and b_1. The equations (1.4.16) for homothety are

$$96(|a_0|^2 + |b_0|^2 + |c_0|^2) + 6(|a_1|^2 + |b_1|^2 + |c_1|^2) = 1,$$

$$\begin{aligned} &48(|a_0|^2+|b_0|^2+|c_0|^2)+12(|a_1|^2+|b_1|^2+|c_1|^2)+t^2=1,\\ &3(a_1^2+b_1^2+c_1^2)+8c_0t=0,\\ &6(a_0\bar{a}_1+b_0\bar{b}_1+c_0\bar{c}_1)-c_1t=0. \end{aligned} \tag{3.6.9}$$

Note that these equations are invariant under the action (3.6.8) of $SO(2)$. For fixed $t \in \mathbf{R}$ we can solve the first two equations and obtain

$$\begin{aligned} |a_0|^2+|b_0|^2+|c_0|^2 &= r_0(t)^2\\ |a_1|^2+|b_1|^2+|c_1|^2 &= r_1(t)^2, \end{aligned}$$

where

$$\begin{aligned} r_0(t)^2 &= \frac{1}{144}(1+4t^2)\\ r_1(t)^2 &= \frac{1}{18}(1-8t^2). \end{aligned}$$

The second equation reduces the range of t to

$$0 \le t \le \frac{1}{\sqrt{8}}.$$

If $t = 1/\sqrt{8}$ then $a_1 = b_1 = c_1 = 0$. The third equation in (3.6.9) gives $c_0 = 0$ (the fourth is automatically satisfied) so that we have

$$|a_0|^2+|b_0|^2 = \frac{1}{96}.$$

We obtain that, for $t = 1/\sqrt{8}$, the solution set is the 3-sphere (of radius $1/\sqrt{96}$). The action of $SO(2)$ on (a_0, b_0) (whose orbits are essentially given by the fibres of the Hopf map) reduces this to a 2-dimensional solution set.

Now let $0 \le t < 1/\sqrt{8}$. Since both radii $r_0(t)$ and $r_1(t)$ are positive, the first two equations above say that

$$(a_0, b_0, c_0) \in S^5_{r_0(t)} \quad \text{and} \quad (a_1, b_1, c_1) \in S^5_{r_1(t)}$$

in two copies of $\mathbf{C}^3$. If $t = 0$ then the third and fourth equations in (3.6.9) reduce to

$$a_1^2+b_1^2+c_1^2 = 0$$

and

$$a_0\bar{a}_1+b_0\bar{b}_1+c_0\bar{c}_1 = 0.$$

The first of these is a complex quadric that intersected with $S^5_{1/\sqrt{18}}$ gives a smooth 3-dimensional manifold for (a_1, b_1, c_1). (In fact, topologically, this is the real projective space.) For fixed (a_1, b_1, c_1), the second equation is a complex plane that when intersected with $S^5_{1/12}$, gives a great 3-sphere. Putting these together, the product is a 6-dimensional manifold on which $SO(2)$ acts without fixed points. The quotient gives a 5-dimensional solution set.

Finally, let $0 < t < 1/\sqrt{8}$. Given $(a_1, b_1, c_1) \in S^5_{r_1(t)}$, we use the third equation in (3.6.9) to get

$$c_0 = -\frac{3}{8t}(a_1^2 + b_1^2 + c_1^2).$$

The fourth equation in (3.6.9) is an affine complex plane

$$a_0\bar{a}_1 + b_0\bar{b}_1 + c_0\bar{c}_1 = \frac{1}{6}c_1 t$$

that, when intersected with $S^5_{r_0(t)}$ and with the value of c_0 known, reduces the solution set for (a_0, b_0, c_0) to at most one dimension. This is because a_1 and b_1 cannot vanish simultaneously. (Indeed, if $a_1 = b_1 = 0$ then $c_0\bar{c}_1 = -3c_1^2\bar{c}_1/(8t) = -3|c_1|^2c_1/(8t)$. On the other hand, $c_0\bar{c}_1 = c_1 t/6$. Combining these, we obtain $t = 0$, a contradiction.) This, combined with the 5-dimensional solution set for (a_1, b_1, c_1) gives a 6-dimensional solution set. As before, the action of $SO(2)$ reduces this to 5-dimensions.

Summarizing, for fixed $0 \leq t \leq 1/\sqrt{8}$, the solution set is always at most 5-dimensional. Varying t now gives (3.6.5).

Next we consider $\mathbf{II}_1$ in the splitting (3.6.4). Given a full minimal immersion $f : S^3 \to S_V$ of type II, if f is linearly nonrigid, i.e. $\dim \mathcal{M}_f \geq 1$, then the points on $\partial\mathcal{M}_f$ correspond to type I minimal immersions. We thus have $\partial\mathcal{M}_f \subset \mathbf{I}$. Thus, to describe $\mathbf{II}_1$ we consider line segments connecting pairs of points in $\mathbf{I}$ and use Theorem 2.3.5 to make sure that the points in the interior of the segment correspond to type II quartic minimal immersions. Since $\mathbf{I}$ is a single orbit through $\langle\mathcal{I}\rangle$, we may assume that one endpoint of the segment is $\langle\mathcal{I}\rangle$. We now choose $g = a + jb \in SU(2)$ with $g' = \gamma g \gamma \in SU(2)'$, and let the other endpoint be $\langle\mathcal{I} \circ g'\rangle$. Again by Theorem 2.3.5, the space of components of any quartic minimal immersion corresponding to an interior point of the segment connecting these two points is the $SU(2)$-module

$$V_{\mathcal{I}\circ g'} + V_{\mathcal{I}}.$$

Assuming that the endpoints are distinct, the interior points correspond to type II or type III according to whether this $SU(2)$-module is $3R_4$ or $4R_4$. To simplify the computations, we consider the quotient

$$(V_{\mathcal{I}\circ g'} + V_{\mathcal{I}})/V_{\mathcal{I}} = V_{\mathcal{I}\circ g'}/(V_{\mathcal{I}\circ g'} \cap V_{\mathcal{I}}).$$

This quotient is trivial iff

$$\langle\mathcal{I} \circ g'\rangle = \langle\mathcal{I}\rangle,$$

a task we also have to carry out to prove Theorem 3.6.8 since g' then belongs to the isotropy group of $SU(2)'$.

The quotient is equal to R_4 or $2R_4$ according to whether we have type II or type III in the aforementioned line segment. Technically speaking, we need to make the substitution $z \mapsto az - \bar{b}\bar{w}$ and $w \mapsto \bar{b}\bar{z} + aw$ corresponding

to $g' = \gamma g \gamma$, $g = a + jb$, in each of the polynomials in

$$V_{\mathcal{I}} = \operatorname{span}\{z^4 - \bar{w}^4, z^2\bar{w}^2, z^3 w + \bar{z}\bar{w}^3, z\bar{z}^2 w - \bar{z}w^2\bar{w}, \\ \Im(z^2 w^2), |z|^4 - 4|z|^2|w|^2 + |w|^4\}$$

(cf. (3.6.2)) and work out the components modulo $V_{\mathcal{I}}$. Elementary computations now give that $V_{\mathcal{I}\circ g'}$ modulo $V_{\mathcal{I}}$ is spanned by the following polynomials:

$$\mu z^4 - 4\beta z^3\bar{w} - 4\bar{\beta} z\bar{w}^3 \tag{3.6.10}$$

$$\nu z^4 + 2\alpha z^3\bar{w} - 2\bar{\alpha} z\bar{w}^3 \tag{3.6.11}$$

$$\mu z^3 w + \beta z^3\bar{z} - \bar{\beta} w\bar{w}^3 - 3\beta z^2 w\bar{w} + 3\bar{\beta} z\bar{z}\bar{w}^2 \tag{3.6.12}$$

$$2\nu z^3 w - \alpha z^3\bar{z} - \bar{\alpha} w\bar{w}^3 + 3\alpha z^2 w\bar{w} + 3\bar{\alpha} z\bar{z}\bar{w}^2 \tag{3.6.13}$$

$$\Im(\mu z^2 w^2) - 4\Im(\beta(zw^2\bar{w} - z^2\bar{z}w)) \tag{3.6.14}$$

$$\Re(\nu z^2 w^2) - 2\Re(\alpha(z^2\bar{z}w - zw^2\bar{w})), \tag{3.6.15}$$

where

$$\begin{aligned} \alpha &= ab(|a|^2 - |b|^2) \\ \beta &= a^3\bar{b} + \bar{a}b^3 \\ \mu &= a^4 - \bar{a}^4 - b^4 + \bar{b}^4 \\ \nu &= a^2b^2 + \bar{a}^2\bar{b}^2. \end{aligned} \tag{3.6.16}$$

Lemma 3.6.10. (3.6.10) – (3.6.15) *are linearly dependent iff*

$$\Re(\alpha\bar{\beta}) = 0 \quad \text{and} \quad \alpha\mu + 2\beta\nu = 0. \tag{3.6.17}$$

PROOF. We first observe that the three pairs of polynomials (3.6.10)-(3.6.11), (3.6.12)-(3.6.13) and (3.6.14)-(3.6.15) are mutually orthogonal. Thus, we need to study the linear dependence of each pair of polynomials. The lemma follows by case-by-case verification, splitting the first two pairs of polynomials into real and imaginary parts and evaluating each 4×4-subdeterminant of the corresponding 4×6-matrices. The last pair gives only 2×2-subdeterminants of a 2×4-matrix.

The remaining task is to work out (3.6.16) in terms of a and b. The first equation in (3.6.16) gives

$$(|a|^2 - |b|^2)\Re(a^2\bar{b}^2) = 0. \tag{3.6.18}$$

It is convenient to use "isoparametric" coordinates on S^3, i.e. to set

$$a = \cos(t)\, e^{i\theta} \quad \text{and} \quad b = \sin(t)\, e^{i\phi} \tag{3.6.19}$$

(cf. Section 1.2). $t = 0$ and $t = \pi/2$ correspond to the two great orthogonal circles cut out from S^3 by the span of the first and last two coordinate axes; $t \in (0, \pi/2)$ corresponds to the Clifford torus T_t parametrized by θ and ϕ.

Case I. Let $|a|^2 = |b|^2$. We are on the "middle" Clifford torus $T_{\pi/4}$. We have $\alpha = 0$ so that (3.6.17) reduces to $\beta\nu = 0$.

If $\beta = 0$ then, substituting (6.3.19) into the expression of β we obtain

$$\phi = \theta + (2k+1)\frac{\pi}{4}, \quad k \in \mathbf{Z},$$

or equivalently,

$$a = \frac{1}{\sqrt{2}} e^{i\theta}, \quad \text{and} \quad b = \frac{1}{\sqrt{2}} e^{i\theta} \epsilon^{2k+1}, \quad k \in \mathbf{Z},$$

where $\epsilon = e^{i\frac{\pi}{4}}$.

If $\nu = 0$, we get

$$\phi = -\theta + (2k+1)\pi/4, \quad k \in \mathbf{Z},$$

so that

$$a = \frac{1}{\sqrt{2}} e^{i\theta} \quad \text{and} \quad b = \frac{1}{\sqrt{2}} e^{-i\theta} \epsilon^{2k+1}, \quad k \in \mathbf{Z}.$$

Summarizing Case I, the solution set is the union of eight closed curves in $T_{\pi/4}$ and they lift to $[0, 2\pi]^2$ to give line segments with slope ± 1 and θ- and ϕ-intercepts being any odd multiples of $\pi/4$ (Figure 26).

Case II. We assume that $t \neq \pi/4$. If $t = 0$ then

$$a = e^{i\theta} \quad \text{and} \quad b = 0.$$

The solution set is the entire great circle T_0.

If $t = \pi/2$ then

$$a = 0 \quad \text{and} \quad b = e^{i\phi},$$

and the solution set is $T_{\pi/2}$.

Finally, let $0 < t < \pi/2$ and $t \neq \pi/4$. Working out the coefficients α, β, μ, ν, and substituting them into the second equation in (3.6.16), we finally arrive at the solution set

$$a = \cos t\, \epsilon^l \epsilon^{2k+1} \quad \text{and} \quad b = \sin t\, \epsilon^l, \quad k, l \in \mathbf{Z}. \tag{3.6.20}$$

For fixed t as above, this is the union of 32 points and on $[0, 2\pi]^2$ they correspond to the intersection points of the straight segments obtained above. As t moves, these points sweep 32 curves that, on $T_{\pi/4}$, meet the existing solution set in triple intersection points, and on T_0 and $T_{\pi/2}$ they also produce 8 triple intersection points distributed equidistantly. (Compare this also with Figure 23 on page 35.) Summarizing, the solution set consists of 26 closed curves meeting in 48 triple intersection points. Looking at each case separately, we see that the triple intersection points are given (as quaternions) by

$$\frac{1}{\sqrt{2}}(\epsilon^k + j\epsilon^l), \; k \not\equiv l \,(\mathrm{mod}\, 2); \;\; \epsilon^k, \;\; j\epsilon^l, \;\; k, l \in \mathbf{Z}. \tag{3.6.21}$$

These form a group of order 48, and this group is conjugate in S^3 to the binary octahedral group $\mathbf{O}^*$ (Theorem 1.2.4). By abuse of notation, we denote this conjugate by the same symbol. We obtain that the orbit $\mathbf{I}$ is the octahedral manifold $S^3/\mathbf{O}^*$. Theorem 3.6.8 follows.

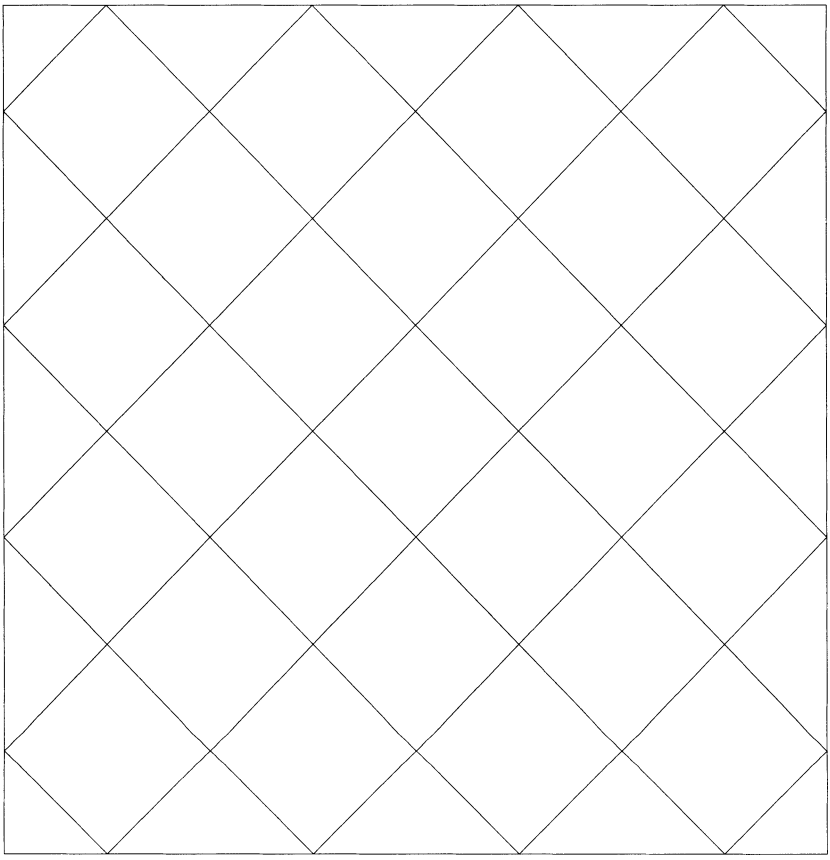


Figure 26.

Looking back now at the 26 curves above, we see that on the quotient $\mathbf{I} = S^3/\mathbf{O}^*$ they give exactly 3 closed curves intersecting at $\langle \mathcal{I} \rangle$. After conjugation with γ, they become orbits of the (mutually orthogonal) 1-parameter subgroups corresponding to Z, $(1/\sqrt{2})(Y+X)$ and $(1/\sqrt{2})(Y-X)$ in $su(2)$, where Z, X, Y is the standard basis of $su(2)$ given in (1.4.12). We denote these orbits by σ, σ' and σ''. More explicitly, σ is parametrized by

$$\theta \mapsto \gamma e^{i\theta}\gamma \cdot \langle \mathcal{I} \rangle, \quad \theta \in \mathbf{R}$$

(corresponding to $t = 0$ in Case II) and σ' (resp. σ'') are parametrized by (3.6.20) with $k = l = 0$ (resp. $k = 1$ and $l = 0$). Note that σ, σ' and σ'' intersect orthogonally at $\langle \mathcal{I} \rangle$.

We now take a closer look at σ. A quick check of Case II reveals that $V_{\mathcal{I}\circ(\gamma e^{i\theta}\gamma)}$, modulo $V_{\mathcal{I}}$, does not depend on θ. The same is true for

$$V_{\mathcal{I}\circ(\gamma e^{i\theta}\gamma)} + V_{\mathcal{I}}$$

so that Theorem 2.3.5 implies that σ is on the boundary of the relative moduli space corresponding to any interior point of any segment connecting two distinct points of σ. We choose the midpoint of the line segment connecting $\langle \mathcal{I} \rangle$ and $e^{i\pi/8} \cdot \langle \mathcal{I} \rangle$ that has the type II representative $\mathcal{J} : S^3 \to S^{14}$ given by

$$\begin{aligned}\mathcal{J}(z,w) = (1/\sqrt{2})\big(&z^4, w^4, 2\sqrt{3}z^2\bar{w}^2, 2z^3w, 2zw^3,\\ &2\sqrt{3}(z\bar{z}^2w - \bar{z}w^2\bar{w}), \sqrt{6}z^2w^2, |z|^4 - 4|z|^2|w|^2 + |w|^4\big).\end{aligned}$$

(The explicit form of $\mathcal{J}$ is obtained by elementary computations in the use of Theorem 2.3.5.) Thus, we have $\sigma \subset \partial\mathcal{M}_{\mathcal{J}}$. The next step is to show that equality holds. For this, we first observe that $\mathcal{J}$ is $U(2)$-equivariant. In fact, we claim that the line segment

$$\mathbf{R} \cdot \langle \mathcal{J} \rangle \cap (\mathcal{M}^4)^{SU(2)} \tag{3.6.22}$$

parametrizes all full quartic $U(2)$-equivariant minimal immersions $f : S^3 \to S_V$. Indeed, the $U(2)$-equivariant quartic minimal immersions are parametrized by the fixed point set $(\mathcal{M}^4)^{U(2)} = (R_8')^{U(2)}$ so that all we need to show is that this is 1-dimensional. Since R_8' is $SU(2)$-fixed, we have $(R_8')^{U(2)} = (R_8')^{\Gamma}$, where

$$\Gamma = \{\, \mathrm{diag}\,(e^{i\theta}, e^{i\theta}) \,|\, \theta \in \mathbf{R}\} \subset SU(2)' \tag{3.6.23}$$

is the center of $U(2)$. As noted above, $\gamma \in O(4)$ switches R_8' and R_8 and Γ to

$$\Gamma' = \{\, \mathrm{diag}\,(e^{i\theta}, e^{-i\theta}) \,|\, \theta \in \mathbf{R}\}; \tag{3.6.24}$$

the standard (1-dimensional) maximal torus in $SU(2)$. Thus, $(R_8')^{\Gamma}$ corresponds to $(R_8)^{\Gamma'}$. On the other hand, Γ' acts on the standard basis in R_8 diagonally with a unique Γ'-fixed polynomial $-z^2w^2$ and the claim follows.

Remark. For $p = 2d$ even, $W_{2d} = \mathcal{H}_2^d$ (for reasons of dimension), where the $SU(2)$-module structure on the space of spherical harmonics on S^2 is given by the projection $SU(2) \to SO(3)$. Thus we also have $R_{2d} = \mathcal{H}_2^d$ as real modules. The $SU(2)'$-orbit of $\langle \mathcal{J} \rangle$ is $\mathbf{R}P^2$ imbedded minimally in its respective 8-sphere as the image of the standard minimal immersion $f_2 : S^2 \to S^8$. Indeed, $(\mathcal{F}^4)^{SU(2)} = R_8' = \mathcal{H}_2^4$ and $(R_8')^{U(2)}$ corresponds to the zonals $(\mathcal{H}_2^4)^{SO(2)}$ whose $SO(3)$-orbit on the unit sphere gives the image of f_2.

We now return to the proof that equality holds in $\sigma \subset \partial \mathcal{M}_{\mathcal{J}}$. Clearly, $\mathcal{M}_{\mathcal{J}}$ is at least 2-dimensional. Since $\langle \mathcal{J} \rangle$ is Γ-fixed, Γ leaves $\mathcal{M}_{\mathcal{J}}$ and its boundary invariant. Γ acts on $\partial \mathcal{M}_{\mathcal{J}}$ without fixed points since a fixed point is automatically $U(2)$-fixed and there are only two of these on the entire boundary. Thus, $\dim \mathcal{M}_{\mathcal{J}}$ must be even, therefore either 2 or 4. Finally, if $\mathcal{M}_{\mathcal{J}}$ were 4-dimensional, its boundary $\partial \mathcal{M}_{\mathcal{J}}$ would be a topological S^3 (by convexity) and it would have to coincide with $\mathbf{I}$ (for reasons of dimension). The latter is $S^3/\mathbf{O}^*$ that is topologically distinct from S^3. We obtain that $\mathcal{M}_{\mathcal{J}}$ is 2-dimensional, and thus it is a flat circular 2-disk $\mathcal{D}$ with $\langle \mathcal{J} \rangle$ being the center.

The argument is entirely analogous for σ' and σ'' so that they are the boundary circles of 2-disks $\mathcal{D}'$ and $\mathcal{D}''$. Note that $\mathcal{D}$, $\mathcal{D}'$, and $\mathcal{D}''$ are orthogonal to each other at the common boundary point $\langle \mathcal{I} \rangle$. We now let $SU(2)'$ act on this configuration and realize that $\mathcal{D}'$ and $\mathcal{D}''$ are on the $SU(2)'$-orbit of $\mathcal{D}$. We thus arrive at (3.6.7).

At this point, without having studied the type III quartic minimal immersions, we have to postpone the proof of (3.6.6). (For a direct proof, see Problem 3.18.)

We now consider type III quartic minimal immersions. We first claim that the antipodal $\mathcal{J}^o$ of $\mathcal{J}$ is of type III. Recall that $\langle \mathcal{J} \rangle$ is the midpoint of the line segment connecting $\langle \mathcal{I} \rangle$ and $\langle \mathcal{I} \circ (\gamma e^{i\pi/8}\gamma) \rangle$ both of type I. Thus, the antipodal of $\langle \mathcal{J} \rangle$ must be on the segment connecting $\langle \mathcal{I}^o \rangle$ and $\langle \mathcal{I} \circ (\gamma e^{i\pi/8}\gamma)^o \rangle$ provided that this segment is on the boundary. Thus, by Theorem 2.3.5 again, we need to work out

$$V_{\mathcal{I}^o} + V_{\mathcal{I} \circ (\gamma e^{i\pi/8}\gamma)^o}.$$

By Theorem 2.3.19, this is equal to

$$V_{\mathcal{I}}^{\perp} + V_{\mathcal{I} \circ (\gamma e^{i\pi/8}\gamma)}^{\perp} = (V_{\mathcal{I}} \cap V_{\mathcal{I} \circ (\gamma e^{i\pi/8}\gamma)})^{\perp}.$$

On the other hand

$$\dim(V_{\mathcal{I}} \cap V_{\mathcal{I} \circ (\gamma e^{i\pi/8}\gamma)}) = \dim V_{\mathcal{I}} + \dim V_{\mathcal{I} \circ (\gamma e^{i\pi/8}\gamma)} - \dim(V_{\mathcal{I}} + V_{\mathcal{I} \circ (\gamma e^{i\pi/8}\gamma)})$$

and this is $10 + 10 - 15 = 5$-dimensional and the claim follows. Thus $\dim V_{\mathcal{J}^o} = 20$ and $\mathcal{J}^o$ is of type III. Summarizing, we see that $\mathcal{J}^o$ is the unique full $U(2)$-equivariant quartic boundary minimal immersion of type III.

Consider the real $SU(2)'$-module R_4'. Since $-I$ acts trivially on R_4', it can be identified with the $SO(3)$-module $\mathcal{H}_2^2$. Recall now from Section 2.7 (cf. the discussion after Corollary 2.7.2) that in the unit sphere $S_{R_4'} = S^4$, the $SO(3)$-orbits form a homogeneous (isoparametric) family of hypersurfaces with two antipodal singular orbits, that are imbedded as Veronese surfaces in S^4. The rest of the orbits are principal. The "middle" principal orbit is minimal and self-antipodal, while the the rest of the principal orbits are paired in antipodal pairs. This last assertion follows from the fact that the orbits are the level hypersurfaces of the (essentially unique) cubic

isoparametric function on S^4. $SO(3)$ also acts on the projective quotient $P_{R_4'} = S_{R_4'}/\{\pm I\} = \mathbf{R}P^4$. This action has a unique singular orbit, and a unique exceptional orbit (whose twofold cover is the middle principal orbit in S^4). The rest of the orbits are principal. The orbit space is thus a line segment with one endpoint corresponding to the unique singular orbit, and the other corresponding to the unique exceptional orbit.

Theorem 3.6.11. III *is everywhere dense, open, and connected in the* 8*-dimensional boundary* $\partial(\mathcal{M}^4)^{SU(2)}$. *For any type III minimal immersion* $f : S^3 \to S_V$, *the relative moduli space* $\mathcal{M}_f$ *is* 4*-dimensional. The quotient* $\mathbf{III}/\mathcal{M}$ *by the open relative moduli (obtained by collapsing the relative moduli in* **III** *to points) is* $SO(3)$*-equivariantly homeomorphic with the projective* 4*-space* $P_{R_4'} = \mathbf{R}P^4$. *The point* $\mathcal{M}_{\mathcal{J}^o}$ *in the quotient* $\mathbf{III}/\mathcal{M}$ *is on the unique singular* $SO(3)$*-orbit.*

Remark. We note here that Theorem 3.6.11 corrects part c) of Theorem C in Toth-Ziller [1], where an error occured in the computation of the dimension of the relative moduli $\mathcal{M}_{\mathcal{J}^o}$.

The proof of Theorem 3.6.11 will be carried out in the rest of this section. First of all, by (3.6.5), the complement of **III** in $\partial(\mathcal{M}^4)^{SU(2)}$ is at least of codimension 2 so that **III** is everywhere dense, open, and connected in $\partial(\mathcal{M}^4)^{SU(2)}$. The first statement of the theorem follows.

The plan for the rest of the proof is as follows. We first construct a set $\mathcal{S}$ of congruence classes of type III minimal immersions, and show that the quotient $\mathcal{S}/\mathcal{M}$ of $\mathcal{S}$, obtained by identifying points in the same open relative moduli, can be parametrized by the real projective plane $\mathbf{R}P^2$. It will also be apparent that $\langle \mathcal{J}^o \rangle \notin \mathcal{S}$.

Then we consider the map

$$\Omega : \mathbf{III} \to P_{R_4'}, \tag{3.6.25}$$

defined as follows: Given a type III minimal immersion $f : S^3 \to S_V$, the orthogonal complement $V_f^\perp$ of the space of components V_f, is an *irreducible* real $SU(2)$-submodule of $\mathcal{H}^4 \cong R_4 \otimes R_4'$. By Lemma 2.7.9

$$V_f^\perp = R_4 \otimes W_f', \tag{3.6.26}$$

where $W_f' \subset R_4'$ is a line. We define

$$\Omega(\langle f \rangle) = W_f'. \tag{3.6.27}$$

Since $\Omega(\langle f \rangle)$ depends only on V_f, it is clear that Ω factors through the canonical projection $\mathbf{III} \to \mathbf{III}/\mathcal{M}$ (where $\mathbf{III}/\mathcal{M}$ is the quotient of **III** by the open relative moduli), and imbeds $\mathbf{III}/\mathcal{M}$ into $P_{R_4'}$. We will prove that Ω is *onto*, so that Ω will induce the $SO(3)$-equivariant homeomorphism between $\mathbf{III}/\mathcal{M}$ and $P_{R_4'}$ in Theorem 3.6.11.

To show surjectivity of Ω, we will first prove that $\mathcal{S} \subset \mathbf{III}$ is mapped by Ω to a projective plane (i.e. a totally geodesic surface) in $P_{R_4'} = \mathbf{R}P^4$.

Since $\langle \mathcal{J}^o \rangle \notin \mathcal{S}$, connectedness of **III** along with a study of the $SO(3)$ orbit structure on $P_{R_4'}$ will imply that Ω is onto.

We will conclude with an easy argument in comparing dimensions which will show that the relative moduli of all type III minimal immersions are 4-dimensional.

Remark. The definition of Ω has been suggested by Weingart [1] (Kapitel 8). Based on the observation that the $SO(3)$-orbits in $P_{R_4'}$ can be parametrized by a line segment, he considered an "angular invariant" associated to each orbit. Here we follow a more geometric path.

The "projective model" $P_{R_4'}$ of the quotient $\mathbf{III}/\mathcal{M}$ is very useful in understanding various incidence relations among all relative moduli. Indeed, we have the obvious extension

$$\Omega : \partial(\mathcal{M}_3^4)^{SU(2)} \to \cup_{n=1}^3 G_n(R_4'),$$

where $G_n(R_4')$ is the Grassmann manifold of the n-dimensional linear subspaces of R_4'. We interpret an element in $G_n(R_4')$ as a projective $(n-1)$-space in $P_{R_4'}$. By Theorem 2.3.5, the image of Ω is closed under intersections. Since, for minimal immersions $f : S^3 \to S_V$ and $f' : S^3 \to S_{V'}$, $\mathcal{M}_{f'} \subset \partial\mathcal{M}_f$ iff $V_{f'} \subset V_f$ with proper inclusion iff $f' \leftarrow f$ with f and f' incongruent, we obtain the following:

Corollary 3.6.12. *Let $f_1 : S^3 \to S_{V_1}$ and $f_2 : S^3 \to S_{V_2}$ be incongruent quartic minimal immersions.*

(a) If f_1 and f_2 are both of type I then there exists a unique relative moduli corresponding to either a linearly nonrigid type II minimal immersion or a type III minimal immersion $f : S^3 \to S_V$ such that $\langle f_1 \rangle, \langle f_2 \rangle \in \partial\mathcal{M}_f$. These two cases are mutually exclusive.

(b) If f_1 and f_2 are both of type II then there is at most one relative moduli of a type III minimal immersion whose boundary contains $\mathcal{M}_{f_1} \cup \mathcal{M}_{f_2}$.

(c) If f_1 and f_2 are both of type III then either $\mathcal{M}_{f_1}$ and $\mathcal{M}_{f_2}$ are disjoint, or $\mathcal{M}_{f_1} \cap \mathcal{M}_{f_2}$ is the relative moduli of a type II minimal immersion.

(d) In general, the set of all relative moduli of type III minimal immersions that contain the relative moduli of a type II minimal immersion can be parametrized by $\mathbf{R}P^1$.

PROOF. All these statements follow from basic facts in projective geometry. (a) Two projective planes in $\mathbf{R}P^4$ intersect in a single point or in a projective line. (b) Two projective lines in $\mathbf{R}P^4$ are either disjoint or meet at a single point. (c) There is a unique projective line that passes through two given projective points. The projective line may or may not correspond to the relative moduli of a type II minimal immersion (see Problem 3.20). (d) A projective line can be parametrized by $\mathbf{R}P^1$.

We now return to the main line. Recall from the study of type II minimal immersions that we considered the space of components

$$V_f = V_{\mathcal{I}} + V_{\mathcal{I}\circ g'} \tag{3.6.28}$$

of a quartic $SU(2)$-equivariant minimal immersion $f : S^3 \to S_V$, where $\langle f\rangle$ is in the interior of the line segment connecting $\langle\mathcal{I}\rangle$ and $\langle\mathcal{I}\circ g'\rangle$, $g' = \gamma g\gamma$, $g = a + jb \in S^3 = SU(2)$. We showed that V_f modulo $V_{\mathcal{I}}$ is spanned by the polynomials (3.6.10)-(3.6.15), and that V_f is of type II iff (3.6.17) with (3.6.16) are satisfied.

To reformulate this last statement, we set

$$\begin{aligned}\xi &= \Re(\alpha\bar\beta),\\ \eta &= \Re(\alpha\mu + 2\beta\nu),\\ \zeta &= \Im(\alpha\mu + 2\beta\nu).\end{aligned} \tag{3.6.29}$$

Then f is of type III iff $(\xi,\eta,\zeta) \neq (0,0,0)$. We call (ξ,η,ζ) the *coordinates* of f. By (3.6.16) and (3.6.29), the coordinates of f are homogeneous degree 8 polynomials in a, b, $\bar a$, $\bar b$.

As a first step in proving Theorem 3.6.11, we consider the set $\mathcal{S}$ (of congruence classes) of type III minimal immersions $f : S^3 \to S_V$ satisfying (3.6.28), and we will show that the quotient $\mathcal{S}/\mathcal{M}$ of this set, obtained by identifying points in the same relative moduli, can be parametrized by the *homogeneous coordinates* $[\xi : \eta : \zeta]$ on $\mathbf{R}P^2$.

Let $f : S^3 \to S_V$ be a type III minimal immersion with $\langle f\rangle \in \mathcal{S}$ and coordinates (ξ,η,ζ). Simple computation in the use of the scalar product (1.4.3) and (3.6.10)-(3.6.15) shows that $V_f^{\perp}$ is spanned by (the real and imaginary parts of) the polynomials

$$\begin{aligned}&\xi z^4 - \frac{\eta}{2}(z^3\bar w - z\bar w^3) - i\frac{\zeta}{2}(z^3\bar w + z\bar w^3),\\ &\xi z^3 w + \frac{\eta}{4}(z^3\bar z - 3z^2 w\bar w + w\bar w^3 - 3z\bar z\bar w^2)\\ &\qquad + i\frac{\zeta}{4}(z^3\bar z - 3z^2 w\bar w - w\bar w^3 + 3z\bar z\bar w^2),\\ &\xi\Re(z^2w^2) - \frac{\eta}{4}\Re(zw^2\bar w - z^2\bar z w) + \frac{\zeta}{4}\Im(zw^2\bar w - z^2\bar z w).\end{aligned} \tag{3.6.30}$$

A quick look at these polynomials shows that $[\xi : \eta : \zeta] \in \mathbf{R}P^2$ uniquely determines the relative moduli of $\langle f\rangle \in \mathcal{S}$.

For the parametrization of $\mathcal{S}/\mathcal{M}$ by $\mathbf{R}P^2$, it remains to be shown that, for each $[\xi : \eta : \zeta] \in \mathbf{R}P^2$, there is a type III minimal immersion $f : S^3 \to S_V$ such that $\langle f\rangle \in \mathcal{S}$ has homogeneous coordinates $[\xi : \eta : \zeta]$. This is the consequence of the following:

Lemma 3.6.13. *The map* $\Pi : \mathbf{C}^2 \to \mathbf{R}\times\mathbf{C} = \mathbf{R}^3$ *defined by*

$$\Pi(a,b) = (\Re(\alpha\bar\beta), \alpha\mu + 2\beta\nu) = (\xi,\eta,\zeta),$$

is surjective.

PROOF. By homogeneity of Π, we can constrain (a, b) to $S^3 \subset \mathbf{C}^2$, and write Π in spherical coordinates as a map $\Pi : S^3 \times \mathbf{R}_+ \to \mathbf{R} \times \mathbf{C}$, where $\mathbf{R}_+$ corresponds to the radial component. Using (3.6.16) and (3.6.29), we obtain

$$\Pi(a, b, r) = r^8\big((|a|^2 - |b|^2)\Re(a^2\bar{b}^2), 2iab\Im(a^4 + b^4) + 4\bar{a}\bar{b}\Re(a^2\bar{b}^2)\big),$$

where $r \in \mathbf{R}_+$ is the radial variable. We rewrite this using the isoparametric coordinates (3.6.19):

$$\begin{aligned}\Pi(t, \theta, \phi, r) &= r^8 \sin(2t) \\ &\times \left(\frac{\sin(4t)}{8}\cos(2(\theta - \phi)), ie^{i(\theta+\phi)}\Im(\cos^4 t \cdot e^{4i\theta} + \sin^4 t \cdot e^{4i\phi})\right. \\ &+ \left.\frac{\sin^2(2t)}{2}e^{-i(\theta+\phi)}\cos(2(\theta - \phi))\right).\end{aligned}$$

It is convenient to make the substitutions

$$u = 2(\theta + \phi) \quad \text{and} \quad v = 2(\theta - \phi)$$

and obtain

$$\begin{aligned}\Pi(t, u, v, r) &= r^8 \sin(2t) \\ &\times \left(\frac{\sin(4t)}{8}\cos v, ie^{iu/2}(\cos^4 t \cdot \sin(u + v) + \sin^4 t \cdot \sin(u - v))\right. \\ &+ \left.\frac{\sin^2(2t)}{2}e^{-iu/2}\cos v\right).\end{aligned}$$

Again by homogeneity, it is enough to show that $\{0\} \times \mathbf{C}$ and $\{1\} \times \mathbf{C}$ are contained in the image of Π. For the first inclusion, we let $v = \pi/2$ and compute

$$\Pi(t, u, \pi/2, r) = r^8\frac{\sin(4t)}{2}ie^{iu/2}\cos u \cdot (0, 1).$$

It is clear that these points cover $\{0\} \times \mathbf{C}$ when t, u, r vary. For the second inclusion, we let $z = \rho e^{i\vartheta}$, $\rho \geq 0$ and $\vartheta \in \mathbf{R}$, and claim that $(1, z)$ is in the image of Π. For this, we set

$$\begin{aligned}t &= \frac{1}{2}\tan^{-1}\left(\frac{\rho}{2}\right), \\ u &= -2\vartheta \\ v &= -\tan^{-1}\left(\frac{\sin^4 t + \cos^4 t}{\cos(2t)}\tan u\right) \\ r &= \left(\frac{8}{\sin(2t)\sin(4t)\cos v}\right)^{1/8}.\end{aligned}$$

Substituting these into the expression of II, the claim follows. This completes the proof of the lemma.

Summarizing, we obtained that the quotient $\mathcal{S}/\mathcal{M}$ can be parametrized by the homogeneous coordinates $[\xi : \eta : \zeta]$, in particular, $\mathcal{S}/\mathcal{M}$ is topologically $\mathbf{R}P^2$.

We will now show that $\langle \mathcal{J}^o \rangle \notin \mathcal{S}$. The space of components of $\mathcal{J}^o$ (complexified) is

$$V_{\mathcal{J}^o} \otimes_{\mathbf{R}} \mathbf{C} = \mathcal{H}^{4,0} \oplus \mathcal{H}^{3,1} \oplus \mathcal{H}^{1,3} \oplus \mathcal{H}^{0,4}. \tag{3.6.31}$$

Indeed, this follows from the way $\mathcal{J}^o$ is defined; both $\mathcal{I}$ and $\mathcal{I}_{\gamma e^{i\pi/8}\gamma}$ have common components in $\mathcal{H}^{2,2}$, say $|z|^4 - 4|z|^2|w|^2 + |w|^2$ (cf. (3.6.2)), so, by Theorem 2.3.19, their antipodals $\mathcal{I}^o$ and $\mathcal{I}^o_{\gamma e^{i\pi/8}\gamma}$ have spaces of components orthogonal to these components. Now, $\langle \mathcal{J}^o \rangle$ is the midpoint of the segment connecting $\langle \mathcal{I}^o \rangle$ and $\langle \mathcal{I}^o_{\gamma e^{i\pi/8}\gamma} \rangle$, and Theorem 2.3.5 applies. In particular, we have

$$V^{\perp}_{\mathcal{J}^o} = \mathcal{H}^{2,2}.$$

Comparing this with (3.6.30), we see that $\langle \mathcal{J}^o \rangle$ is not contained in $\mathcal{S}$.

Recall that $\Omega|_{\mathcal{S}}$ factors through the canonical projection $\mathcal{S} \to \mathcal{S}/\mathcal{M}$, and imbeds $\mathcal{S}/\mathcal{M}$ into $P_{R'_4}$. We now must show that the image $\Omega(\mathcal{S})$ is a projective plane in $P_{R'_4}$. To do this, recall that, for each $(\xi, \eta, \zeta) \neq (0, 0, 0)$, the linear span of the polynomials in (3.6.30) is $R_4 \otimes W'_f$, where $W'_f \subset R'_4$ is a line, and f has coordinates (ξ, η, ζ). Splitting the complex polynomials in (3.6.30) into real and imaginary parts, the resulting five polynomials can be written as

$$u_i(\xi, \eta, \zeta) \otimes p(\xi, \eta, \zeta), \quad i = 1, \ldots, 5, \tag{3.6.32}$$

where

$$\operatorname{span}\{u_i(\xi, \eta, \zeta) \mid i = 1, \ldots, 5\} = R_4$$

(for any $(\xi, \eta, \zeta) \neq 0$), and $W'_f = \mathbf{R} \cdot p(\xi, \eta, \zeta)$. By (3.6.32), u_i and p are unique only up to mutual scaling, and they can also be chosen to depend on (ξ, η, ζ) continuously, at least locally. By definition, the tensor products in (3.6.32) depend on (ξ, η, ζ) *linearly*, so that we have

$$u_i(X) \otimes p(X) + u_i(X') \otimes p(X') = u_i(X + X') \otimes p(X + X'), \ i = 1, \ldots, 5, \tag{3.6.33}$$

where $X = (\xi, \eta, \zeta), X' = (\xi', \eta', \zeta')$, and $X + X' = (\xi + \xi', \eta + \eta', \zeta + \zeta')$ are all nonzero. Since $\{u_k(X + X')\}_{k=1}^5 \subset R_4$ is a basis, we have the equations

$$\begin{aligned} u_i(X) &= \sum_{k=1}^{5} c_{ik}(X, X') u_k(X + X'), \\ u_i(X') &= \sum_{k=1}^{5} c_{ik}(X', X) u_k(X + X'). \end{aligned} \tag{3.6.34}$$

Substituting these into (3.6.33), we obtain

$$\sum_{k=1}^{5} u_k(X+X') \otimes \big(c_{ik}(X,X')p(X) + c_{ik}(X',X)p(X')\big) \\ = u_i(X+X') \otimes p(X+X').$$

Thus, we have

$$c_{ik}(X,X')p(X) + c_{ik}(X',X)p(X') = 0, 0 \le i \ne k \le 5, \tag{3.6.35}$$
$$c_{ii}(X,X')p(X) + c_{ii}(X',X)p(X') = p(X+X'), \quad i = 1,\dots,5.$$

To study the projective linear structure of the image $\Omega(\mathcal{S})$, i.e. the set of all lines $W'_f = \mathbf{R}\cdot p(X)$, $X = (\xi,\eta,\zeta)$, where $\langle f\rangle$ varies on $\mathcal{S}$, we may assume that $p(X)$ and $p(X')$ are linearly independent. Then the first equation implies that $c_{ik}(X,X') = c_{ik}(X',X) = 0$ for $i \ne k$. The second equation implies that $c_{ii}(X,X')$ and $c_{ii}(X',X)$ are independent of i, so that we can suppress the indices. Using these in (3.6.33) and (3.6.34), we obtain

$$u_i(X) = c(X,X')u_i(X+X'),$$
$$u_i(X') = c(X',X)u_i(X+X'),$$
$$c(X,X')p(X) + c(X',X)p(X') = p(X+X').$$

Projective linearity of the image of $\mathcal{S}$ under Ω follows. Hence $\Omega(\mathcal{S})$ is a projective plane in $P_{R'_4}$.

To complete the proof of Theorem 3.6.11, we need to show that Ω : **III** $\to P_{R'_4}$ is onto. Since **III** is connected and Ω is $SO(3)$-equivariant, it is enough to show that the unique singular orbit and the unique exceptional orbit both contain points from the image of Ω. (Recall that these two orbits correspond to the two extremal points of the segment $P_{R'_4}/SO(3)$.) $\langle\mathcal{J}^o\rangle \in$ **III** is mapped by Ω to the singular orbit $P_{R'_4}$ since $\langle\mathcal{J}^o\rangle$ has 1-dimensional isotropy. It remains to show that $\Omega(\mathcal{S})$ intersects the exceptional orbit. It is easier to see this in the twofold cover $S_{R'_4} = S^4$, where the middle $SO(3)$-orbit (in the homogeneous isoparametric family) doubly covers the exceptional orbit. Since $\Omega(\mathcal{S})$ is a projective plane in $P_{R'_4}$, it is doubly covered by a great 2-sphere in $S_{R'_4}$. These two double covers intersect since any great sphere in $S_{R'_4}$ is self-antipodal, and the middle $SO(3)$-orbit is the zero-set of the cubic isoparametric function that defines the $SO(3)$-orbits in $P_{R'_4}$. With the exception of the dimensionality of the relative moduli, Theorem 3.6.11 follows.

Finally, we prove that all relative moduli of type III minimal immersions are 4-dimensional. By Proposition 3.6.7, they are at least 4-dimensional. To show that this lower bound is sharp, we first consider the case $f = \mathcal{J}^o$. Since $\mathcal{M}_{\mathcal{J}^o}$ is $U(2)$ invariant, the center $\Gamma \subset SU(2)'$ of $U(2)$ acts on $\partial\mathcal{M}_{\mathcal{J}^o}$ without fixed points. It follows that $\dim\mathcal{M}_{\mathcal{J}^o}$ is even. Assume that $\dim\mathcal{M}_{\mathcal{J}^o} \ge 6$. By Theorem 2.3.8, the $SU(2)'$ orbit of $\operatorname{int}\mathcal{M}_{\mathcal{J}_o}$ is an 8-dimensional smooth manifold. This is because Γ leaves $\mathcal{M}_{\mathcal{J}_o}$ invariant, and

$SU(2)'$ does not have 2-dimensional subgroups. The set $SU(2)'(\operatorname{int}\mathcal{M}_{\mathcal{J}_o})$ must then be open in $\partial(\mathcal{M}^4)^{SU(2)}$. Since its boundary is contained in **I**∪**II**, it follows that $SU(2)'(\operatorname{int}\mathcal{M}_{\mathcal{J}_o})$ must be a connected component of **III**. But **III** is connected, so that $SU(2)'(\operatorname{int}\mathcal{M}_{\mathcal{J}_o})$ must be the whole of **III**. This contradicts the existence of the subset $\mathcal{S}$ constructed in the first step.

Notice that this proof applies to any type III minimal immersion $f : S^3 \to S_V$ with $\langle f\rangle \in SU(2)'(\operatorname{int}\mathcal{M}_{\mathcal{J}_o})$.

We now assume that $f : S^3 \to S_V$ is a type III minimal immersion with $\langle f\rangle \notin SU(2)'(\operatorname{int}\mathcal{M}_{\mathcal{J}_o})$. This means that the isotropies on $\mathcal{M}_f$ are finite (since otherwise the centroid of $\mathcal{M}_f$ would have 4-dimensional isotropy, thereby conjugate to $U(2)$, in contradiction with the uniqueness of $\mathcal{J}^o$). Assuming that $\dim\mathcal{M}_f \geq 5$, Theorem 2.3.8 applies again, and we obtain that $\dim SU(2)'(\operatorname{int}\mathcal{M}_f) \geq 8$. As before, this must be a connected component of **III**, therefore the whole of **III**. This contradicts to the existence of $\mathcal{J}^o$. Theorem 3.6.11 follows.

Remark. It is instructive to compare the results of this section with those of Section 2.7, particularly, to look for analogies in the structures of $(\mathcal{L}^2_3)^{SU(2)}$ and $(\mathcal{M}^4_3)^{SU(2)}$.

3.7 Additional Topic: The Inverse of Ψ

A symmetric multilinear map $h : \mathcal{H}^1 \times \ldots \times \mathcal{H}^1 \to \mathcal{H}^{2l}$ with $2k$ arguments, $k \geq l \geq 1$, can be thought of as a linear map $h : S^{2k}(\mathcal{H}^1) \to \mathcal{H}^{2l}$ by restriction from the Weyl's space $\otimes^{2k}\mathcal{H}^1$ to the space of symmetric tensors $S^{2k}(\mathcal{H}^1)$. On the other hand, $S^{2k}(\mathcal{H}^1)$ is isomorphic with $\mathcal{P}^{2k}$, the space of $(2k)$-homogeneous polynomials on $\mathbf{R}^{m+1}$. Thus, h can be viewed as a linear map $h : \mathcal{P}^{2k} \to \mathcal{H}^{2l}$.

We now assume that h is traceless, i.e. contraction of h viewed as a tensor with respect to any two indices is zero. This is equivalent to the vanishing of h on $\mathcal{P}^{2(k-1)}\rho^2$. Restricting h to the orthogonal complement $\mathcal{H}^{2k} \subset \mathcal{P}^{2k}$ (cf. (2.1.1)), we obtain a linear map $h : \mathcal{H}^{2k} \to \mathcal{H}^{2l}$, or equivalently, a tensor $h \in \mathcal{H}^{2l} \otimes (\mathcal{H}^{2k})^* \cong \mathcal{H}^{2l} \otimes \mathcal{H}^{2k}$.

By (3.1.7), $\mathcal{H}^{2l} \otimes \mathcal{H}^{2k}$ decomposes as

$$\mathcal{H}^{2l} \otimes \mathcal{H}^{2k} = \sum_{(u,v)\in\triangle_0^{2l,2k};\, u+v \text{ even}} V^{(u,v,0,\ldots,0)}.$$

The northern vertex $(2l, 2k)$ of $\triangle_0^{2l,2k}$ corresponds to the $SO(m+1)$-module $V^{(2l,2k,0,\ldots,0)} \subset \mathcal{H}^{2l} \otimes \mathcal{H}^{2k}$. Since this is also a component of $S^2(\mathcal{H}^p)$ for $p \geq k + l$, and a component of $\mathcal{E}^p$ $(k \geq 1)$ and $\mathcal{F}^p$ $(k \geq 2)$, it is important to recognize when h does belong to $V^{(2l,2k,0,\ldots,0)}$. In this section we give a necessary and sufficient condition for $h : \mathcal{H}^1 \times \ldots \times \mathcal{H}^1 \to \mathcal{H}^{2l}$ to belong to $V^{(2l,2k,0,\ldots,0)}$. The explicit formula for h as an element in $S^2(\mathcal{H}^p)$ also provides an explicit expression for the inverse of Ψ^k_p.

To formulate our main result, in addition to the operator $D : \mathcal{H}^p \otimes \mathcal{H}^q \to \mathcal{H}^{p-1} \otimes \mathcal{H}^{q-1}$ introduced in Section 2.6, we need the following homomorphism

$$\iota : \mathcal{H}^p \otimes \mathcal{H}^q \to \mathcal{H}^{p-1} \otimes \mathcal{H}^{q+1}$$

of $SO(m+1)$-modules defined by

$$\iota = \sum_{i=0}^{m} \partial_i \otimes \delta_i,$$

where $\delta_a = H(a^* \cdot)$, $a \in \mathbf{R}^{m+1}$, and $\delta_i = \delta_{e_i}$, with $\{e_i\}_{i=0}^m \subset \mathbf{R}^{m+1}$ the standard basis (Section 2.1). Up to a constant multiple, ι is a generalization of the homomorphism $\iota_+ : \mathcal{H}^p \to \mathcal{H}^{p-1} \otimes \mathcal{H}^1$, $q = 0$, introduced in Section 2.1.

Theorem 3.7.1. *Given a traceless symmetric $(2k)$-multilinear map $h : \mathcal{H}^1 \times \ldots \times \mathcal{H}^1 \to \mathcal{H}^{2l}$, h defines an element in the component $V^{(2l,2k,0,\ldots,0)}$ of $\mathcal{H}^{2l} \otimes \mathcal{H}^{2k}$ iff*

$$h(x,\ldots)(x) = \sum_{i=0}^{m} x_i h(e_i,\ldots)(x) = 0, \tag{3.7.1}$$

where $\{e_i\}_{i=0}^m \subset \mathbf{R}^{m+1}$ is the standard basis. Given h satisfying this, $(D^\top)^{p-(l+k)}\iota^{l-k}h$ is an element in $V^{(2l,2k,0,\ldots,0)} \subset S^2(\mathcal{H}^p)$ whose Ψ_p^k-image is, up to a constant multiple, h.

For the proof we need some preparation. First recall that differentiating of the harmonic projection formula (2.1.8) gives

$$\partial_a H(b^*\chi) = \langle a,b\rangle\chi + b^*\partial_a\chi - \frac{4p}{\lambda_{2p}} a^*\partial_b\chi - \frac{2p}{\lambda_{2p}}\partial_a\partial_b\chi\cdot\rho^2,\ \chi \in \mathcal{H}^p.$$

As in Section 2.1, we introduce the operator (acting on $\mathcal{H}^p$):

$$E_{ab} = b^*\partial_a - a^*\partial_b, \quad a,b \in \mathbf{R}^{m+1}.$$

With this, the formula above is written as

$$\partial_a H(b^*\chi) = \langle a,b\rangle\chi + E_{ab}\chi + \left(1 - \frac{4p}{\lambda_{2p}}\right) H(a^*\partial_b\chi),\ \chi \in \mathcal{H}^p.$$

Notice that the multiple of ρ^2 cancels as each term is harmonic. More concisely, we have on $\mathcal{H}^p$:

$$\partial_a \circ \delta_b = \langle a,b\rangle I + E_{ab} + \left(1 - \frac{4p}{\lambda_{2p}}\right)\delta_a \circ \partial_b.$$

Lemma 3.7.2. *Up to a constant multiple, D and ι commute, i. e., on $\mathcal{H}^p \otimes \mathcal{H}^q$, we have*

$$D \circ \iota = \left(1 - \frac{4p}{\lambda_{2p}}\right)(\iota \circ D).$$

PROOF. On $\mathcal{H}^p \otimes \mathcal{H}^q$, we have

$$\begin{aligned} D \circ \iota &= \sum_{i,k=0}^{m} \partial_k \partial_i \otimes \partial_k \delta_i \\ &= \sum_{i,k=0}^{m} \partial_k \partial_i \otimes \left(\delta_{ki} + E_{ki} + \left(1 - \frac{4p}{\lambda_{2p}}\right) \delta_k \partial_i \right) \\ &= \left(1 - \frac{4p}{\lambda_{2p}}\right) \sum_{i,k=0}^{m} \partial_k \partial_i \otimes \delta_k \partial_i \\ &= \left(1 - \frac{4p}{\lambda_{2p}}\right) (\iota \circ D). \end{aligned}$$

Lemma 3.7.3. *On $\mathcal{H}^p \otimes \mathcal{H}^q$, we have*

$$\begin{aligned} \frac{\mu_q}{\mu_{p-1}} (\iota^\top \circ \iota) - \frac{\mu_{q-1}}{\mu_p} (\iota \circ \iota^\top) &= (p-q) I \\ &+ \frac{2}{\mu_{p-1}\mu_{q-1}} \left(\frac{2p}{\lambda_{2p}} - \frac{2q}{\lambda_{2q}} \right) D^\top \circ D, \end{aligned} \tag{3.7.2}$$

where μ_p is given by (2.1.24).

PROOF. By Lemma 2.1.4, on $\mathcal{H}^p \otimes \mathcal{H}^q$, we have

$$\iota^\top = \frac{\mu_{p-1}}{\mu_q} \sum_{i=0}^{m} \delta_i \otimes \partial_i,$$

and

$$D^\top = \mu_{p-1}\mu_{q-1} \sum_{i=0}^{m} \delta_i \otimes \delta_i.$$

Thus, we obtain

$$\iota \circ \iota^\top = \frac{\mu_p}{\mu_{q-1}} \iota \left(\sum_{i=0}^{m} \delta_i \otimes \partial_i \right) = \frac{\mu_p}{\mu_{q-1}} \sum_{i,k=0}^{m} \partial_k \delta_i \otimes \delta_k \partial_i.$$

We continue as

$$\begin{aligned} \frac{\mu_{q-1}}{\mu_p} (\iota \circ \iota^\top) &= \sum_{i,k=0}^{m} \left(\delta_{ki} + E_{ki} + \left(1 - \frac{4p}{\lambda_{2p}}\right) \delta_k \partial_i \right) \otimes \delta_k \partial_i \\ &= \sum_{i=0}^{m} I \otimes \delta_i \partial_i + \sum_{i,k=0}^{m} E_{ki} \otimes \delta_k \partial_i \\ &\quad + \left(1 - \frac{4p}{\lambda_{2p}}\right) \sum_{i,k=0}^{m} \delta_k \partial_i \otimes \delta_k \partial_i \\ &= qI - \frac{1}{2} E + \frac{1}{\mu_{p-1}\mu_{q-1}} \left(1 - \frac{4p}{\lambda_{2p}}\right) D^\top \circ D, \end{aligned}$$

where, after the second equality, we replaced $\delta_k\partial_i$ by $(1/2)(\delta_k\partial_i - \delta_i\partial_k)$ (using skew symmetry of E_{ki}), and introduced the second order differential operator

$$E = \sum_{i,k=0}^{m} E_{ki} \otimes E_{ki}.$$

Summarizing, we have on $\mathcal{H}^p \otimes \mathcal{H}^q$:

$$\frac{\mu_{q-1}}{\mu_p}(\iota \circ \iota^\top) = q\,I - \frac{1}{2}E + \frac{1}{\mu_{p-1}\mu_{q-1}}\left(1 - \frac{4p}{\lambda_{2p}}\right)D^\top \circ D. \tag{3.7.3}$$

The computation of $\iota^\top \circ \iota$ is simpler and we get

$$\frac{\mu_q}{\mu_{p-1}}(\iota^\top \circ \iota) = p\,I - \frac{1}{2}E + \frac{1}{\mu_{p-1}\mu_{q-1}}\left(1 - \frac{4q}{\lambda_{2q}}\right)D^\top \circ D. \tag{3.7.4}$$

Subtracting (3.7.4) from (3.7.3), we arrive at (3.7.2).

Theorem 3.7.4. *For $p > q$, the homomorphism*

$$\iota : \mathcal{H}^p \otimes \mathcal{H}^q \to \mathcal{H}^{p-1} \otimes \mathcal{H}^{q+1}$$

is injective.

PROOF. We show injectivity by induction. Assume that ι is injective on $\mathcal{H}^{p-1} \otimes \mathcal{H}^{q-1}$. Since D is onto (Corollary A.3.3 in Appendix 3) and, up to a constant multiple, it commutes with ι (Lemma 3.7.2), ι is injective on $(\ker D)^\perp \subset \mathcal{H}^p \otimes \mathcal{H}^q$. On the other hand, ι is injective on

$$\ker D = \sum_{r=0}^{q} V^{(p+q-r,r,0,\ldots,0)} \subset \mathcal{H}^p \otimes \mathcal{H}^q.$$

Indeed, assume that ι is zero on $V^{(p+q-r,r,0,\ldots,0)}$. Since $\iota^\top$ maps $V^{(p+q-r,r,0,\ldots,0)}$ into itself we obtain that $\iota \circ \iota^\top$ is zero on $V^{(p+q-r,r,0,\ldots,0)}$. This contradicts Lemma 3.7.3.

PROOF OF THEOREM 3.7.1. $h \in \mathcal{H}^{2l} \otimes \mathcal{H}^{2k}$ belongs to $V^{(2l,2k,0,\ldots,0)}$ iff $D(h) = 0$ and h is orthogonal to $\mathcal{H}^{2l+1} \otimes \mathcal{H}^{2k-1}$. We write the orthogonality condition as $\iota^\top(h) = 0$, where $\mathcal{H}^{2l+1} \otimes \mathcal{H}^{2k-1}$ is realized in $\mathcal{H}^{2l} \otimes \mathcal{H}^{2k}$ as the image of ι. In coordinates, we have

$$h = \sum_{i_1,\ldots,i_{2k}=0}^{m} h(e_{i_1},\ldots,e_{i_{2k}}) \otimes H(y_{i_1}\cdots y_{i_{2k}}) \in \mathcal{H}^{2l} \otimes \mathcal{H}^{2k}.$$

Since h is traceless, the harmonic projection can be dispensed with

$$h = \sum_{i_1,\ldots,i_{2k}=0}^{m} h(e_{i_1},\ldots,e_{i_{2k}}) \otimes y_{i_1}\cdots y_{i_{2k}}.$$

The condition $D(h) = 0$ translates into

$$D(h) = \sum_{i=0}^{m} \sum_{i_1,\ldots,i_{2k}=0}^{m} \partial_i h(e_{i_1}, \ldots, e_{i_{2k}}) \otimes \partial_i(y_{i_1} \cdots y_{i_{2k}})$$
$$= 2k \sum_{i=0}^{m} \sum_{i_2,\ldots,i_{2k}=0}^{m} \partial_i h(e_i, e_{i_2}, \cdots, e_{i_{2k}}) \otimes y_{i_2} \ldots y_{i_{2k}} = 0.$$

This holds iff

$$\sum_{i=0}^{m} \partial_i h(e_i, \ldots) = 0. \tag{3.7.5}$$

The orthogonality condition $\iota^\top(h) = 0$ rewrites as

$$\iota^\top(h) = \sum_{i=0}^{m} \sum_{i_1,\ldots,i_{2k}=0}^{m} H(x_i h(e_{i_1}, \ldots, e_{i_{2k}})) \otimes \partial_i(y_{i_1} \cdots y_{i_{2k}})$$
$$= 2k \sum_{i=0}^{m} \sum_{i_2,\ldots,i_{2k}=0}^{m} H(x_i h(e_i, e_{i_2}, \ldots, e_{i_{2k}})) \otimes y_{i_2} \cdots y_{i_{2k}} = 0.$$

This holds iff

$$\sum_{i=0}^{m} H(x_i h(e_i, \ldots)) = 0. \tag{3.7.6}$$

Since h takes values in $\mathcal{H}^{2l}$, we have

$$H(x_i h(e_i, \ldots)) = x_i h(e_i, \ldots) - \frac{\rho^2}{4l + m - 1} \partial_i h(e_i, \ldots).$$

Thus the conditions (3.7.5) and (3.7.6) give

$$\sum_{i=0}^{m} x_i h(e_i, \ldots) = 0.$$

The first statement follows. The second statement follows from Schur's lemma.

Problems

3.1. Show that, for $p \geq 6$ even, the system of equations (1.4.16) is solvable. Conclude that there exist full minimal immersions $f : S^3 \to S_{R_p}$ of degree $p \geq 6$.

3.2. Use (3.2.6) and homogeneity to show that

$$\sum_{i,k=0}^{m} x_i x_k \Psi(C)(e_i, e_k)(x) = p^2 \Psi^0(C)(x),$$

where $\{e_i\}_{i=0}^m \subset \mathbf{R}^{m+1}$ is the standard orthonormal basis. Give another proof of Proposition 3.1.2 (first for $f = f_p$ and then in general).

3.3. Show, by direct computation, that

$$\triangle^{p-1}\Psi(C)(a,b) = 0, \quad C \in \mathcal{E}^p.$$

Conclude that in the canonical decomposition of $\Psi(C)(a,b) \in \mathcal{P}^{2(p-1)}$ there is no constant term (multiplied by $\rho^{2(p-1)}$).

3.4. Use the method in the proof of Theorem 2.5.3 to show that Ψ, viewed as a linear map $\Psi : \mathcal{E}^p \otimes S_0^2(\mathbf{R}^{m+1}) \to \mathcal{P}^{2(p-1)}$, is onto.

Problems 3.5-3.8 outline a different final step in proving Theorem 3.1.6.

3.5. Let $m = 2m_0 + 1$ be odd. Derive the congruence

$$\Psi(f)(e_0, e_1) \equiv -2\sum_{j=0}^{n_0} \Im\left(\frac{\partial f^j}{\partial z_0}\frac{\partial \bar{f}^j}{\partial z_0}\right) (\mathrm{mod}\, \rho^4)$$

for complex spherical harmonic p-forms $f : \mathbf{C}^{m_0+1} \to \mathbf{C}^{n_0+1}$, where f is written in terms of complex variables $z_0, \bar{z}_0, \ldots, z_{m_0}, \bar{z}_{m_0}$, and $e_0 = (1,0,\ldots,0), e_1 = (i,0,\ldots,0) \in \mathbf{C}^{m_0+1}$. (Note that in the congruence the holomorphic and antiholomorphic components cancel.)

3.6. Fill in the gaps in the following argument to show that, for $m = 2m_0+1$ odd and $p = 2q$ even, $q \geq 2$, there exists a full p-eigenmap $f : S^{2m_0+1} \to S^{2\binom{m+p}{p}-3}$, such that, in the decomposition

$$\Psi(\langle f \rangle)(e_0, e_1) = \sum_{l=1}^{p-1} h^l \rho^{2(p-l-1)}$$

we have

$$h^{p-2} \neq 0 \quad \text{and} \quad h^{p-1} \neq 0.$$

Here $h^l = h^l(e_0, e_1)$ is the l-th canonical coefficient of f (Theorem 3.2.7).

Consider the complex Veronese map $Ver^{\mathbf{C}}_{m_0,p} : S^{2m_0+1} \to S^{2\binom{m+p}{p}-1}$, given by

$$Ver^{\mathbf{C}}_{m_0,p}(x) = \left(\sqrt{\frac{p!}{i_0!\ldots i_m!}} z_0^{i_0} \ldots z_m^{i_m}\right)_{i_0+\ldots+i_m=p;\, i_0,\ldots,i_m \geq 0}.$$

Let $p = 2q$ be even. Replace the three components

$$\sqrt{\frac{(2q)!}{(q-1)!(q+1)!}} z_0^{q-1} z_1^{q+1}, \quad \sqrt{\frac{(2q)!}{(q-1)!(q+1)!}} z_0^{q+1} z_1^{q-1}, \quad \frac{\sqrt{(2q)!}}{q!} z_0^q z_1^q$$

of $Ver^{\mathbf{C}}_{m_0,p}$ by the two components

$$\sqrt{\frac{(2q)!}{(q-1)!(q+1)!}} (|z_0|^2 - |z_1|^2) z_0^{q-1} z_1^{q-1}, \quad \sqrt{\frac{3q+1}{q+1}} \frac{\sqrt{(2q)!}}{q!} z_0^q z_1^q.$$

Verify that the resulting map is spherical so that we obtain a full p-eigenmap $f : S^{2m_0+1} \to S^{2\binom{m_0+p}{p}-3}$. Use Problem 3.5 to show that

$$\Psi(f)(e_0, e_1) \equiv \frac{2(2q)!}{(q-1)!(q+1)!} \Im(q\xi_{q-1,q-1} - (q-1)\xi_{q-2,q}) (\mathrm{mod}\, \rho^4),$$

where

$$\xi_{k,l}(z) = z_0^2|z_0|^{2k}|z_1|^{2l},\ k,l \geq 0.$$

Use the harmonic projection formula (2.1.7) along with the complex form of the Laplacian to derive

$$\begin{aligned}\frac{(q+1)!(q-1)!}{2(2q)!}\Psi(f)(e_0,e_1) \equiv\ & qH(\Im\xi_{q-1,q-1}) - (q-1)H(\Im\xi_{q-2,q}) \\ & + \frac{q(q-1)}{4(q-1)+m_0+1}\Big(H(\Im\xi_{q-2,q-1}) \\ & + (q-1)H(\Im\xi_{q-1,q-2}) \\ & - (q-2)H(\Im\xi_{q-3,q})\Big)\ (\mathrm{mod}\,\rho^4).\end{aligned}$$

Finally, show that

$$qH(\Im\xi_{q-1,q-1}) - (q-1)H(\Im\xi_{q-2,q}) \neq 0,$$

and

$$H(\Im\xi_{q-2,q-1}) + (q-1)H(\Im\xi_{q-1,q-2}) - (q-2)H(\Im\xi_{q-3,q}) \neq 0.$$

3.7. Let $m = 2(m_0+1)$ be even. Use the domain-dimension-raising operator to obtain examples from the ones in Problem 3.6. Let $f : S^{2m_0+1} \to S^n$, $n = 2\binom{2m_0+p+1}{p} - 3$, be a full p-eigenmap, $p = 2q$, $q \geq 2$, as in Problem 3.6 and let $\tilde{f} : S^{2m_0+2} \to S^{n+N(2m_0+2,p)-N(2m_0+1,p)}$ be the p-eigenmap obtained from f by raising the domain dimension. Apply Lemma 3.4.1 to prove that

$$\tilde{h}^{p-2} \neq 0 \quad \text{and} \quad \tilde{h}^{p-1} \neq 0,$$

where $\tilde{h}^l(a,b) \in \mathcal{H}^{2l}_{m+1}$, $l = 1,\ldots,p-1$, denote the l-th canonical coefficient of $\tilde{f}$ (evaluated at $(a,b) \in \mathcal{H}^1_m \times \mathcal{H}^1_m$).

3.8. Fill in the gaps in the following argument to give another elementary construction for eigenmaps with even dimensional domain and with the required nonvanishing properties (which does not use the domain-dimension-raising operator). Consider eigenmaps whose components are complex valued spherical harmonics (of real or complex variables) with the first four variables x_0, x_1, x_2, x_3 singled out. Rewrite these in terms of $z_0 = x_0 + ix_1$ and $z_1 = x_2 + ix_3$ and their conjugates.

(a) Show that, for each $m = 2(m_0+1)$ and $p = 2q$ even, there exists a full p-eigenmap $F : S^{2(m_0+1)} \to S^n$ which contains (a constant multiple of)

$$z_0^{q-1}z_1^{q+1} \text{ and } z_0^{q+1}z_1^{q-1}.$$

(Hint: Use induction with respect to q. For $q = 1$, define $F : S^{2(m_0+1)} \to S^n$ by

$$\begin{aligned}F(z,t) = \Big(& z_0^2,\ldots,z_{m_0}^2,(\sqrt{2}z_iz_k)_{0\leq i<k\leq m_0}, \\ & \sqrt{2+2/(m+1)}\,tz_0,\ldots,\sqrt{2+2/(m+1)}\,tz_{m_0}, \\ & t^2 - (|z_0|^2+\ldots+|z_{m_0}|^2)/(m_0+1)\Big),\end{aligned}$$

where $z = (z_0, \ldots, z_{m_0}) \in \mathbf{C}^{m_0+1}$ and $t \in \mathbf{R}$. For the general induction step, assume that, for fixed q, an eigenmap F with two of its coordinates as above exists. Raise the degree twice and show that, up to a constant multiple, $(F^+)^+$ contains

$$H(x_r H(x_s z_0^{q\pm1} z_1^{q\mp1})) = H(x_r x_s z_0^{q\pm1} z_1^{q\mp1}), \quad r, s = 0, 1, 2, 3.$$

Modify this new eigenmap to arrive at one that contains $z_0^{q-1} z_1^{q+1}$ and $z_0^{q+1} z_1^{q-1}$. (In general, if ψ' and ψ'' are components of an eigenmap, then replacing them by $(1/\sqrt{2})(\psi' + \psi'')$ and $(1/\sqrt{2})(\psi' - \psi'')$, or by $(1/\sqrt{2})(\psi' + i\psi'')$ and $(1/\sqrt{2})(\psi' - i\psi'')$ gives new eigenmaps.))

(b) Restart with $F : S^{2(m_0+1)} \to S^n$ as above and replace the two components above by

$$(|z_0|^2 - |z_1|^2) z_0^{q-1} z_1^{q-1} \text{ and } z_0^q z_1^q.$$

with suitable constant multiples. Consider the difference

$$\Psi(f)(e_0, e_1) - \Psi(F)(e_0, e_1).$$

Conclude that (again up to a constant multiple) this difference has nonvanishing harmonic coefficients h^{p-1} and h^{p-2}.

3.9. Generalize Π^0 (Section 2.5) and Π (Section 3.2) to obtain a homomorphism

$$\Pi^k_{p,q} : \mathcal{P}^p \otimes \mathcal{P}^q \to \mathcal{P}^{p+q-2k} \otimes \mathcal{P}^{2k}, \quad 0 \le k \le p, q.$$

Show that

$$\Pi^0_{p+q-2k,2k} \circ \Pi^0_{p,q} = \frac{p!}{(p-k)!} \frac{q!}{(q-k)!} \Pi^0_{p,q}.$$

3.10. Give a short argument why the degree-raising and -lowering operators do not commute.

3.11. Show the following analogue of the Braid lemma: Let P, R be finite dimensional vector spaces and $\mathcal{B} : R \times R \times R \times R \times R \to P$ a quintilinear map that is skew symmetric when the first-third, and second-fourth variables are simultaneously interchanged, and symmetric in the last three variables. Then $\mathcal{B}$ vanishes identically. (Hint: Consider the permutation $[(13)(24)(345)]^5$.) Derive the following rigidity: Let $f : S^m \to S_V$ be a full isotropic minimal immersion of degree p and order of isotropy 2. If $p \le 5$ then f is congruent to f_p.

3.12. Let $f : S^m \to S_V$ be a minimal immersion of degree p and set $f = A \circ f_p$. Show that

$$A \circ \beta_l(f_p)(X^{a_1}, \ldots, X^{a_{2l}})^\sim = \beta_l(f)(X^{a_1}, \ldots, X^{a_{2l}})^\sim, \ 1 \le l \le p.$$

Conclude that A maps the osculating space $\check{\mathcal{O}}^l_{f_p;x}$ of f_p onto the osculating space $\check{\mathcal{O}}^l_{f;x}$ of f (Appendix 2). Assume that f is isotropic of order k and show that A gives rise to an isomorphism between $\mathcal{O}^l_{f_p}$ and $\mathcal{O}^l_f$, $1 \le l \le k$. Use the $SO(m+1)$-equivariance of f_p to verify that $\mathcal{O}^l_{f_p;o} \cong \mathcal{H}^l_{m-1}$, $l = 1, \ldots, p$, $o = (1, 0, \ldots, 0)$, as $SO(m)$-modules. Finally, for $f : S^m \to S_V$ isotropic of order k, derive the lower estimate

$$\dim V \ge \sum_{l=0}^{k} \dim \mathcal{H}^l_{m-1} = \dim \mathcal{H}^k_{m-1}.$$

3.13. Generalize Problem 3.4 to show that Ψ_p^k, viewed as a linear map $\Psi_p^k : \mathcal{F}^{p;k-1} \otimes (\mathcal{H}^{2k})^* \to \mathcal{P}^{2(p-k)}$, is onto.

3.14. Use Problem 2.19 (d) to show that $(\Phi_5^- \circ \Phi_4^+)(\mathcal{M}^4) \subset \operatorname{int} \mathcal{M}^4$. Generalize this to isotropic minimal immersions.

3.15. Follow the argument in the proof of Theorem 2.7.8 to derive Proposition 3.6.7 using (3.1.4).

3.16. Prove nonexistence of an $SU(2)$-equivariant quartic minimal immersion $f : S^3 \to S_V$ with $V = R_4$ using the following argument. Assume that f exists. Notice that the orbits of $SU(2)$ on $S_{R_4} = S^4$ form a homogeneous family of hypersurfaces (cf. the argument after Corollary 2.7.2). Show that the image of f is in the middle minimal hypersurface of this family. Now use the classical fact (Lawson [1]) that the metric induced on S^3 by f is not the constant curvature metric (actually, not even naturally reductive with respect to $SO(4)$).

3.17. Explain the connection between the following two facts: (1) $\mathcal{I}$ realizes a minimal imbedding of the prism manifold S^3/D_2^* into S^9. (2) The $SU(2)$-orbit of $\langle \mathcal{I} \rangle$ is the octahedral manifold $S^3/\mathbf{O}^*$.

3.18. Use the implicit function theorem for the system (3.6.9) to verify (3.6.6).

3.19. Use $U(2)$-equivariance of $\mathcal{J}$ along with the argument in the proof of Theorem 3.6.11 to show that $\dim \mathcal{M}_{\mathcal{J}} \geq 2$.

3.20. Explain why $\Omega : \mathbf{II} \to G_2(R_4')$ cannot be onto.

3.21. Let $G \subset SO(m+1)$ be a closed subgroup. Explain the difference between $(\mathcal{M}^p)^G = \mathcal{M}^p \cap S^2(\mathcal{H}^p)^G$ and $\mathcal{M}^p \cap S^2((\mathcal{H}^p)^G)$, where $S^2((\mathcal{H}^p)^G)$ is considered as a linear subspace of $S^2(\mathcal{H}^p)$ via the inclusion $(\mathcal{H}^p)^G \subset \mathcal{H}^p$. (Note: The examples in Section 1.5 correspond to points in $(\mathcal{M}_3^p)^{SU(2)} \cap S^2((\mathcal{H}_3^p)^{G^*})$, where $G^* \subset SU(2)$ is a finite subgroup.)

3.22. Verify that $\sum_{i=0}^m x_i \Psi(f)(e_i, .) = 0$ using (3.2.3). Confirm that condition (3.7.1) is necessary in Theorem 3.7.1.

4

Lower Bounds on the Range of Spherical Minimal Immersions

4.1 Infinitesimal Rotations of Eigenmaps

In this section, we define a new operator acting on eigenmaps, the *operator of infinitesimal rotations*. The name comes from the fact that this operator associates to a p-eigenmap $f : S^m \to S_V$ another p-eigenmap $\hat{f} : S^m \to S_{V \otimes so(m+1)^*}$, where the components of $\hat{f}$ are obtained by rotating infinitesimally the components of f in each coordinate plane of $\mathbf{R}^{m+1}$. We will study the self-map on the moduli $\mathcal{L}^p$ defined by the correspondence $\langle f \rangle \mapsto \langle \hat{f} \rangle$. It turns out that this self-map is the restriction of a symmetric $SO(m+1)$-module endomorphism of $\mathcal{E}^p$ that can be expressed in terms of the Casimir operator in a simple manner. In view of later applications to $SU(2)$-equivariant eigenmaps and for greater generality, we will define the operator of infinitesimal rotations for an arbitrary closed subgroup $G \subset SO(m+1)$ acting transitively on S^m.

Let $G \subset SO(m+1)$ be a closed subgroup with Lie algebra $\mathcal{G}$, and assume that G acts transitively on S^m. As usual, for each $X \in \mathcal{G}$, we let X also denote the vector field induced on S^m. For a p-eigenmap $f : S^m \to S_V$, we define $\hat{f} : S^m \to V \otimes \mathcal{G}^*$ by

$$\hat{f}(x)(X) = \frac{1}{\sqrt{\lambda_p}} X_x(f), \quad X \in \mathcal{G},\ x \in S^m.$$

Once and for all, we endow the Lie algebra $\mathcal{G}$ with the biinvariant Riemannian metric that induces the standard (curvature one) Riemannian metric on S^m. Given an orthonormal basis $\{X_s\}_{s=1}^n \subset \mathcal{G}$ with dual basis

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$\{\phi_s\}_{s=1}^n \subset \mathcal{G}^*$, the definition of $\hat{f}$ can also be written as

$$\hat{f} = \frac{1}{\sqrt{\lambda_p}} \sum_{s=1}^{n} X_s(f) \otimes \phi_s. \tag{4.1.1}$$

Note that $\hat{f}$ may not be full even if f is. As usual, we restrict a non full map to the linear span of its image and denote the full restriction by the same symbol.

Recall from Section 2.3 that the fixed point set $(\mathcal{L}^p)^G$ of G on $\mathcal{L}^p$ parametrizes the congruence classes of full G-equivariant p-eigenmaps $f : S^m \to S_V$.

Theorem 4.1.1. *(i) For a p-eigenmap $f : S^m \to S_V$, $\hat{f} : S^m \to S_{V \otimes \mathcal{G}^*}$ is spherical, and hence a p-eigenmap.*

(ii) The correspondence $\langle f \rangle \mapsto \langle \hat{f} \rangle$ gives rise to a self-map of $\mathcal{L}^p$ which is the restriction of a symmetric linear map $\mathcal{A}_p = \mathcal{A}_{m,p} : \mathcal{E}^p \to \mathcal{E}^p$.

(iii) All eigenvalues of $\mathcal{A}_p$ are real and contained in $[-1, 1]$.

(iv) $\mathcal{A}_p$ is an endomorphism of the G-module $\mathcal{E}^p|_G$. The eigenspace of $\mathcal{A}_p$ corresponding to the eigenvalue $+1$ is the fixed point set $(\mathcal{E}^p)^G$. The eigenspace of $\mathcal{A}_p$ corresponding to the eigenvalue -1 is contained in the orthogonal complement of $(\mathcal{E}_m^p)^G$ in $(\mathcal{E}_m^p)^{[G,G]}$. In particular, -1 is not an eigenvalue if G is semisimple.

We call $\mathcal{A}_p : \mathcal{L}^p \to \mathcal{L}^p$, $\mathcal{A}_p(\langle f \rangle) = \langle \hat{f} \rangle$, the *operator of infinitesimal rotations.* (For $G = SO(m+1)$, the standard orthonormal basis $\{E_{ik}\}_{0 \le i < k \le m} \subset so(m+1)$ has typical element $E_{ik} = x_k \partial_i - x_i \partial_k$, and this is the infinitesimal rotation in the $x_i x_k$-coordinate plane (cf. Problem 2.5).)

For the proof of Theorem 4.1.1. we need some preparation.

Let $f : S^m \to S_V$ be a p-eigenmap. Since each $X \in \mathcal{G}$ acts as an infinitesimal isometry on S^m, the components of $\hat{f}$ are spherical harmonics of order p on S^m, i.e. $V_{\hat{f}} \subset \mathcal{H}^p$. To prove that $\hat{f}$ maps into the unit sphere of $V \otimes \mathcal{G}^*$, we fix an orthonormal basis $\{X_s\}_{s=1}^n \subset \mathcal{G}$, write $\hat{f}$ as in (4.1.1) and use the well-known fact (Wallach [1]) that

$$\triangle^{S^m} = -\sum_{s=1}^{n} X_s^2.$$

We compute

$$\begin{aligned} \triangle^{S^m} |f|^2 &= 2\langle \triangle^{S^m} f, f \rangle - 2 \sum_{s=1}^{n} |X_s(f)|^2 \\ &= 2\lambda_p |f|^2 - 2 \sum_{s=1}^{n} |X_s(f)|^2, \end{aligned}$$

where we used $\triangle^{S^m} f = \lambda_p f$. This vanishes since f is spherical. Thus, we have

$$|\hat{f}|^2 = \frac{1}{\lambda_p} \sum_{s=1}^{n} |X_s(f)|^2 = |f|^2 = 1.$$

(i) of Theorem 4.1.1 follows.

To obtain an explicit formula for the correspondence $f \mapsto \hat{f}$ on the moduli space $\mathcal{L}^p$, we define the linear map

$$\alpha_p : \mathcal{H}^p \to \mathcal{H}^p \otimes \mathcal{G}^*$$

by

$$\alpha_p(\chi)(X) = \frac{1}{\sqrt{\lambda_p}} X(\chi), \quad \chi \in \mathcal{H}^p, \ X \in \mathcal{G}.$$

In coordinates, we have

$$\alpha_p(\chi) = \frac{1}{\sqrt{\lambda_p}} \sum_{s=1}^{n} X_s(\chi) \otimes \phi_s.$$

Lemma 4.1.2. *$\alpha_p : \mathcal{H}^p \to \mathcal{H}^p \otimes \mathcal{G}^*$ is a linear isometric imbedding:*

$$\alpha_p^\top \circ \alpha_p = I.$$

PROOF. Being an infinitesimal isometry, the action of $X \in \mathcal{G}$ on $\mathcal{H}^p$ is skew-symmetric. Hence the transpose $\alpha_p^\top$ is given by

$$\alpha_p^\top(\chi \otimes \phi_s) = -\frac{1}{\sqrt{\lambda_p}} X_s(\chi), \quad \chi \in \mathcal{H}^p, \ s = 1, \ldots, n.$$

We use this to compute

$$\begin{aligned}(\alpha_p^\top \circ \alpha_p)(\chi) &= \alpha_p^\top \left(\frac{1}{\sqrt{\lambda_p}} \sum_{s=1}^{n} X_s(\chi) \otimes \phi_s \right) \\ &= -\frac{1}{\lambda_p} \sum_{s=1}^{n} X_s^2 \chi = \frac{1}{\lambda_p} \triangle^{S^m} \chi = \chi.\end{aligned}$$

Lemma 4.1.2 follows.

Next we claim that $\hat{f}_p$ (made full) is congruent to f_p.

Lemma 4.1.3. *We have*

$$\hat{f}_p(x) = \alpha_p(f_p(x)), \quad x \in S^m.$$

PROOF. Using the explicit form (2.2.1) of the standard minimal immersion, we compute

$$\hat{f}_p(x) = \frac{1}{\sqrt{\lambda_p}} \sum_{s=1}^{n} X_s \left(\sum_{j=0}^{N(p)} f_p^j(x) f_p^j \right) \otimes \phi_s$$

$$= \frac{1}{\sqrt{\lambda_p}} \sum_{s=1}^{n} \sum_{j=0}^{N(p)} f_p^j(x) X_s(f_p^j) \otimes \phi_s$$
$$= \sum_{j=0}^{N(p)} f_p^j(x) \alpha_p(f_p^j) = \alpha_p(f_p(x)).$$

The lemma follows.

Let $f : S^m \to S_V$ be a full p-eigenmap with $\langle f \rangle = A^\top A - I \in \mathcal{L}^p$, where $A : \mathcal{H}^p \to V$ is the surjective linear map defined by $f = A \circ f_p$. By Lemma 4.1.3, we have

$$\hat{f} = (A \otimes I) \circ \hat{f}_p = (A \otimes I) \circ \alpha_p \circ f_p,$$

where $I = I_{\mathcal{G}^*}$. Using Lemma 4.1.2, we compute

$$\begin{aligned} \langle \hat{f} \rangle &= \alpha_p^\top (A^\top \otimes I)(A \otimes I) \alpha_p - I \\ &= \alpha_p^\top ((A^\top A - I) \otimes I) \alpha_p \\ &= \alpha_p^\top (\langle f \rangle \otimes I) \alpha_p. \end{aligned}$$

In view of this, replacing $\langle f \rangle$ by a symmetric endomorphism of $\mathcal{H}^p$, we define

$$\mathcal{A}_p : S^2(\mathcal{H}^p) \to S^2(\mathcal{H}^p)$$

by

$$\mathcal{A}_p(C) = \alpha_p^\top \circ (C \otimes I) \circ \alpha_p, \quad C \in S^2(\mathcal{H}^p).$$

This is then an extension of $\langle f \rangle \mapsto \langle \hat{f} \rangle$, i.e., we have

$$\mathcal{A}_p(\langle f \rangle) = \langle \hat{f} \rangle.$$

Since $\mathcal{L}^p$ is $\mathcal{A}_p$-invariant ((i) of Theorem 4.1.1 already have been proved), we obtain that $\mathcal{E}^p$ is $\mathcal{A}_p$-invariant. To prove (ii) of Theorem 4.1.1 it remains for us to show the following:

Lemma 4.1.4. *The linear map $\mathcal{A}_p : S^2(\mathcal{H}^p) \to S^2(\mathcal{H}^p)$ is symmetric.*

PROOF. This is a straightforward computation. Given $C, C' \in S^2(\mathcal{H}^p)$ and an orthonormal basis $\{f_p^j\}_{j=0}^{N(p)} \subset \mathcal{H}^p$, we compute

$$\begin{aligned} \langle \mathcal{A}_p(C), C' \rangle &= \sum_{j=0}^{N(p)} \langle \mathcal{A}_p(C) f_p^j, C' f_p^j \rangle \\ &= \sum_{j=0}^{N(p)} \langle (C \otimes I) \alpha_p(f_p^j), \alpha_p(C' f_p^j) \rangle \\ &= \frac{1}{\lambda_p} \sum_{s=1}^{n} \sum_{j=0}^{N(p)} \langle C(X_s f_p^j), X_s(C' f_p^j) \rangle \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{\lambda_p}\sum_{s=1}^{n}\sum_{l=0}^{N(p)}\langle Cf_p^l, X_s(C'(X_s f_p^l))\rangle \\
&= \frac{1}{\lambda_p}\sum_{s=1}^{s}\sum_{l=0}^{N(p)}\langle X_s C f_p^l, C'(X_s f_p^l)\rangle \\
&= \sum_{l=0}^{N(p)}\langle \alpha_p(Cf_p^l), (C'\otimes I)\alpha_p(f_p^l)\rangle \\
&= \sum_{l=0}^{N(p)}\langle Cf_p^l, \mathcal{A}_p(C')f_p^l\rangle \\
&= \langle C, \mathcal{A}_p(C')\rangle.
\end{aligned}
$$

Here we used the fact that infinitesimal isometries are skew on $\mathcal{H}^p$, and replaced $X_s f_p^j$ with $-\sum_{l=0}^{N(p)}\langle f_p^j, X_s f_p^l\rangle f_p^l$. Lemma 4.1.4 and (ii) of Theorem 4.1.1. follow.

Being symmetric on $\mathcal{H}^p$, $\mathcal{A}_p$ is diagonalizable and has real eigenvalues. Since $\mathcal{E}^p$ is $\mathcal{A}_p$-invariant, the same holds on $\mathcal{E}^p$. In addition, since $\mathcal{L}^p \subset \mathcal{E}^p$ is compact (Corollary 2.3.3) and $\mathcal{A}^p$-invariant, all eigenvalues of $\mathcal{A}_p|_{\mathcal{E}^p}$ are contained in $[-1,1]$. We obtain (iii) of Theorem 4.1.1.

We now claim that $\alpha_p : \mathcal{H}^p \to \mathcal{H}^p \otimes \mathcal{G}^*$ is a homomorphism of G-modules, where the G-module structure on $\mathcal{H}^p$ is given by restriction (from $SO(m+1)$ to G), and on $\mathcal{G}$ by the adjoint representation. This is the first statement in (iv) of Theorem 4.1.1. Indeed, for $g \in G$, $\chi \in \mathcal{H}^p$ and $X \in \mathcal{G}$, we have

$$
\begin{aligned}
\alpha_p(g\cdot\chi)(X) &= \frac{1}{\sqrt{\lambda_p}} X(\chi \circ g^{-1}) \\
&= \frac{1}{\sqrt{\lambda_p}} (\operatorname{ad}(g^{-1})X)(\chi)\circ g^{-1} \\
&= g\cdot\alpha_p(\chi)\,(\operatorname{ad}(g^{-1})X).
\end{aligned}
$$

Here in the second equality we used

$$
X_x(\chi\circ g^{-1}) = (\operatorname{ad}(g^{-1})X)_{g^{-1}(x)}\chi, \quad x \in S^m,
$$

a straightforward consequence of the definition of the adjoint representation. In fact, if $t \mapsto \phi_t$ denotes the 1-parameter group of transformations induced by X, we have

$$
\begin{aligned}
X_x(\chi\circ g^{-1}) &= \frac{d}{dt}(\chi\circ g^{-1}\circ\phi_t^{-1})|_{t=0}(x) \\
&= \frac{d}{dt}(\chi\circ(g^{-1}\circ\phi_t\circ g)^{-1})|_{t=0}(g^{-1}(x)) \\
&= (\operatorname{ad}(g^{-1})X)_{g^{-1}(x)}\chi.
\end{aligned}
$$

The claim follows.

Lemma 4.1.5. *$(\mathcal{E}^p)^G$ is the $+1$ eigenspace of $\mathcal{A}_p$.*

PROOF. We first claim that $\mathcal{A}_p$ is the identity map on $(\mathcal{E}^p)^G$. Indeed, as a homomorphism of G-modules, $\mathcal{A}_p$ certainly leaves $(\mathcal{E}^p)^G$ invariant, so that, by Lemma 4.1.4, we need to show that the only eigenvalue of $\mathcal{A}_p$ on $(\mathcal{E}^p)^G$ is 1. To do this, we let $C \in (\mathcal{E}^p)^G$ be an eigenvector of $\mathcal{A}_p$ with eigenvalue $\lambda \in [-1, 1]$. We may assume that $C = \langle f \rangle \in \partial\mathcal{L}^p$, where $f : S^m \to S_V$ is a full p-eigenmap of boundary type. Since $\langle f \rangle \in (\mathcal{E}^p)^G$, f is equivariant with respect to a homomorphism $\rho_f : G \to SO(V)$:

$$f \circ g = \rho_f(g) \circ f, \quad g \in G. \tag{4.1.2}$$

Substituting a one-parameter subgroup $g = \exp(tX)$, $t \in \mathbf{R}$, $X \in \mathcal{G}$, into (4.1.2), and differentiating at $t = 0$, we get

$$X(f) = R_f(X) \circ f,$$

where $R_f : \mathcal{G} \to so(V)$ is the differential of ρ_f. In terms of $\hat{f}$, this is rewritten as

$$\hat{f} = \frac{1}{\sqrt{\lambda_p}} R_f \circ f,$$

where R_f is viewed as a linear map $R_f : V \to V \otimes \mathcal{G}^*$. This shows, in particular, that $V_{\hat{f}} \subset V_f$ for the spaces of components. Since f is of boundary type, so is $\hat{f}$. On the other hand, by assumption, $\langle \hat{f} \rangle = \mathcal{A}_p(\langle f \rangle) = \lambda \langle f \rangle$. Since $\langle f \rangle, \langle \hat{f} \rangle \in \partial\mathcal{L}^p$, this is possible only if $\lambda = 1$ or $\lambda < 0$. The latter means that f and $\hat{f}$ are antipodal. This contradicts Theorem 2.3.5 since $V_f + V_{\hat{f}} = V_f$, and this cannot be the whole of $\mathcal{H}^p$. Thus $\lambda = 1$ and the first claim therefore follows.

For the converse, we assume, on the contrary, that $(\mathcal{E}^p)^G$ is *properly* contained in the $+1$ eigenspace of $\mathcal{A}_p$. Let $f : S^m \to S_V$ be a full p-eigenmap of boundary type that represents an eigenvector $\langle f \rangle$ of $\mathcal{A}_p$ with eigenvalue $+1$ such that $\langle f \rangle \perp (\mathcal{E}^p)^G$. Since $\langle \hat{f} \rangle = \langle f \rangle$, there exists a linear homothety $R : V \to V \otimes \mathcal{G}^*$ satisfying

$$X(f) = R(X) \circ f, \quad X \in \mathcal{G}.$$

This shows that the space of components $V_f = V_{\hat{f}}$ is a $\mathcal{G}$-submodule of $\mathcal{H}^p$. Equivalently, $V_f \subset \mathcal{H}^p$ is a G-submodule of $\mathcal{H}^p$.

We now imitate the definition of the standard minimal immersion for V_f instead of $\mathcal{H}^p$. We let $\{f_0^l\}_{l=0}^n \subset V_f$, $n+1 = \dim V_f$, an orthonormal basis, and define $f_0 : S^m \to V_f$ by

$$f_0(x) = \sum_{l=0}^{n} f_0^l(x) f_0^l.$$

Clearly, f_0 is G-equivariant, and, since G acts transitively on S^m, up to scaling, f_0 maps S^m to the unit sphere of V_f. After scaling, we arrive at a

full p-eigenmap $f_0 : S^m \to S_{V_f}$. By the very definition of f_0, $V_{f_0} = V_f$ but $\langle f_0 \rangle \neq \langle f \rangle$. In addition, since f_0 is G-equivariant, $\langle f_0 \rangle$ is fixed by G.

Consider the half-line with endpoint at $\langle f \rangle$ passing through $\langle f_0 \rangle$. Since $\mathcal{L}^p$ is compact and convex, this half-line intersects $\mathcal{L}^p$ in a finite segment with an endpoint beyond $\langle f_0 \rangle$. (This is because $\langle f_0 \rangle \in \operatorname{int} \mathcal{M}_f$ as $V_f = V_{f_0}$.) We let this endpoint be represented by a full p-eigenmap $f' : S^m \to S_{V'}$. Since f and f_0 have the same range dimension, $\dim V' < \dim V$; in particular, f_0 and f' are incongruent.

Let C denote the centroid of the G-orbit through $\langle f \rangle$. Clearly, C is a fixed point of G. Also, since $\langle f \rangle$ is orthogonal to the G-module $(\mathcal{E}^p)^G$, so is C. This is possible only for $C = 0$.

Now let C' be the centroid of the G-orbit through $\langle f' \rangle$. As before, $C' \in (\mathcal{E}^p)^G$. Moreover, by convexity, $C' \in \mathcal{L}^p$. Since $\langle f_0 \rangle$ is G-fixed and is between $\langle f \rangle$ and $\langle f' \rangle$, the point $\langle f_0 \rangle$ is between $C = 0$ and C' in $\mathcal{L}^p$. Thus, f_0 cannot be of boundary type. This is a contradiction, and the lemma follows.

Finally, the last statement in (iv) of Theorem 4.1.1 will follow from the next result:

Lemma 4.1.6. *The $+1$ eigenspace of $(\mathcal{A}_p)^2$ is contained in $(\mathcal{E}^p)^{[G,G]}$, where $[G,G]$ is the commutator subgroup of G.*

PROOF. Let $C \in \partial\mathcal{L}^p$ be the fixed point of $(\mathcal{A}_p)^2$, and $f : S^m \to S_V$ a full p-eigenmap of boundary type that represents C. By construction, V_f is invariant under the second-order differential operators XY, $X, Y \in \mathcal{G}$. In particular, V_f is a $[\mathcal{G},\mathcal{G}]$-submodule of $\mathcal{H}^p$. As in the proof of Lemma 4.1.5, V_f is a $[G,G]$-submodule. It is well-known that if G acts transitively on S^m, so does $[G,G]$ (cf. Borel [1]). The previous proof (of Lemma 4.1.5) now applies.

4.2 Infinitesimal Rotations and the Casimir Operator

One of our aims in this section is to show that homothety and, in general, isotropy are preserved by the operator $\mathcal{A}_p$, i.e. if a p-eigenmap $f : S^m \to S_V$ is isotropic of order k, $k \geq 1$, then so is $\hat{f}$. In view of Theorems 3.1.6 and 3.5.5, for $G = SO(m+1)$, this follows immediately since $\mathcal{E}^p$ has multiplicity-one decomposition into irreducible $SO(m+1)$-modules. The general case ($G \subset SO(m+1)$ is a closed subgroup transitive on S^m) is more difficult.

In addition, we will establish a simple relation between the restriction $\mathcal{A}_p|_{\mathcal{E}^p}$ and the Casimir operator for $\mathcal{E}^p$ as a G-module. This, for $G = SO(m+1)$, will result an explicit form of the eigenvalues of $\mathcal{A}_p$ on the irreducible components of $\mathcal{E}^p$.

Theorem 4.2.1. *Let $G \subset SO(m+1)$ be a closed subgroup with Lie algebra $\mathcal{G}$, and assume that G acts transitively on S^m. If $f : S^m \to S_V$ is an isotropic minimal immersion of degree p and order of isotropy k, then so is $\hat{f} : S^m \to S_{V \otimes \mathcal{G}^*}$. Equivalently, $\mathcal{A}_p$ maps the moduli $\mathcal{F}_m^{p;k}$ (in (3.5.15)) into itself, $k = 0, \ldots, [p/2] - 1$.*

We have

$$\mathcal{A}_p|_{\mathcal{E}^p} = I - \frac{1}{2\lambda_p} Cas, \tag{4.2.1}$$

where $Cas = -\text{trace}\{(X,Y) \to [X,[Y,.]]\}$ is the Casimir operator of G acting on $\mathcal{E}^p$. For $G = SO(m+1)$ and $m \geq 4$, the eigenvalue $\Lambda_p^{u,v}$ of $\mathcal{A}_p$ on the irreducible component $V^{(u,v,0,\ldots,0)} \subset \mathcal{E}^p$ in (3.1.9) (i.e., $(u,v) \in \triangle_1^p$ and u, v, even) is

$$\Lambda_p^{u,v} = 1 - \frac{\mu^{u,v}}{2\lambda_p}, \tag{4.2.2}$$

where

$$\mu^{u,v} = u^2 + v^2 + u(m-1) + v(m-3) \tag{4.2.3}$$

is the eigenvalue of the Casimir operator on $V^{(u,v,0,\ldots,0)}$.

Remark 1. For $m = 3$, $V^{(u,v)}$, is the sum of two irreducible $SO(4)$-modules (Proposition 3.6.1), and their restrictions to $SU(2)$ are expressible in terms of the irreducible representations $W_{u \pm v}$. It is an elementary fact that the Casimir operator on W_p is the Laplacian, so that the Casimir eigenvalue on W_p is $p(p+2)$. In Section 4.5 we will study this case in detail.

Remark 2. (4.2.2)-(4.2.3) follow from (4.2.1) by representation theory. In fact, the eigenvalue of the Casimir operator on an irreducible representation in terms of the highest weight and the Cartan matrix is known. (For a detailed account, see the article of Wang-Ziller [1]. Note that the terminology in this work is different; Wang and Ziller write a representation in terms of its dominant weight as an integral linear combination of dominant fundamental weights.) In our proof of Theorem 4.2.1 below, (4.2.2)-(4.2.3) will follow *as a byproduct* of the fairly involved proof of the preservance of isotropy under $\mathcal{A}_p$, which is the first statement in Theorem 4.2.1. This way, we obtain an independent proof for the Casimir eigenvalue formula (4.2.3). Theorem 4.2.1, and most of the developments in this chapter are taken from Toth [1,2].

Both proving that $\mathcal{A}_p$ preserves isotropy and the computations leading to the explicit form (4.2.2)-(4.2.3) of the eigenvalues require the same technique (to be expounded in this section). On the other hand, to derive the expression (4.2.1) of $\mathcal{A}_p$ in terms of the Casimir operator is much simpler. In fact, for $C \in S^2(\mathcal{H}^p)$ and $\chi \in \mathcal{H}^p$, using the notations of the previous section, we have

$$\mathcal{A}_p(C)\chi = \alpha_p^\top (C \otimes I)\alpha_p(\chi)$$

$$= \alpha_p^\top (C \otimes I)\left(\frac{1}{\sqrt{\lambda_p}} \sum_{s=1}^{n} X_s \chi \otimes \phi_s\right)$$

$$= \frac{1}{\sqrt{\lambda_p}} \sum_{s=1}^{n} \alpha_p^\top (C X_s \chi \otimes \phi_s)$$

$$= -\frac{1}{\lambda_p} \sum_{s=1}^{n} X_s C X_s \chi.$$

On the other hand, by the definition of the Casimir operator, we have

$$Cas\,(C) = -\sum_{s=1}^{n} [X_s, [X_s, C]]$$

$$= -\sum_{s=1}^{n} X_s^2 \circ C - C \circ \sum_{s=1}^{n} X_s^2 + 2 \sum_{s=1}^{n} X_s \circ C \circ X_s.$$

Now $\sum_{s=1}^{n} X_s^2 = -\triangle^{S^m} = -\lambda_p I$ on $\mathcal{H}^p$ so that the first two terms on the right-hand side give $2\lambda_p I$. By the explicit expansion for $\mathcal{A}_p$ above, the last term is rewritten as $2\sum_{s=1}^{n} X_s C X_s = -2\lambda_p \mathcal{A}_p(C)$. With these (4.2.1) and, hence, (4.2.2) follow.

To complete the proof of Theorem 4.2.1 we need to show the preservance of isotropy under $\mathcal{A}_p$ and (4.2.3). We need several preparatory lemmas.

Lemma 4.2.2. *For $a \in \mathbf{R}^{m+1}$ and $X \in \mathcal{G}$, we have*

$$[\partial_a, X] = \partial_{X \cdot a}. \tag{4.2.4}$$

On the left-hand side, X is the vector field induced by the action of G on $\mathbf{R}^{m+1}$, and on the right-hand side X is the skew-symmetric matrix in $\mathcal{G} \subset so(m+1)$ acting on vectors in $\mathbf{R}^{m+1}$ by matrix multiplication.

PROOF. With the different meanings for X, we have

$$X_x = \partial_{X \cdot x} = \sum_{i=0}^{m} x_i \partial_{X \cdot e_i},$$

where, as usual, $\{e_i\}_{i=0}^{m} \subset \mathbf{R}^{m+1}$ is the standard basis. Using this, we compute

$$[\partial_a, X] = \sum_{i=0}^{m} [\partial_a, x_i \partial_{X \cdot e_i}] = \sum_{i=0}^{m} \partial_a(x_i) \partial_{X \cdot e_i}$$

$$= \sum_{i=0}^{m} a_i \partial_{X \cdot e_i} = \partial_{X \cdot a}.$$

The lemma follows.

Lemma 4.2.3. *For $a \in \mathbf{R}^{m+1}$ and $X, Y \in \mathcal{G}$, we have*

$$X \partial_{Ya} + Y \partial_{Xa} = -\partial_{XYa} + \partial_a YX - YX\partial_a. \tag{4.2.5}$$

PROOF. (4.2.4) with a replaced by Ya, reads as:

$$[X, \partial_{Ya}] = -\partial_{XYa}.$$

Expanding the Lie bracket and using (4.2.4) repeatedly, we compute

$$\begin{aligned} X\partial_{Ya} + \partial_{XYa} &= \partial_{Ya}X = [\partial_a, Y]X \\ &= \partial_a YX - Y\partial_a X \\ &= \partial_a YX - Y([\partial_a, X] + X\partial_a) \\ &= \partial_a YX - Y\partial_{Xa} - YX\partial_a. \end{aligned}$$

The lemma follows.

We now take the trace of both sides of (4.2.5). Since the trace of the bilinear map $(X, Y) \mapsto XY$ on $\mathcal{H}^p$ is $-\triangle^{S^m} = -\lambda_p \cdot I$, and the Casimir operator $Cas\,(a)$ of $a \in \mathcal{G}$ on $\mathbf{R}^{m+1}$ is the trace of the bilinear map $(X, Y) \to -XYa$, we obtain on $\mathcal{H}^p$:

$$2\,\mathrm{trace}\,\{(X, Y) \mapsto X\partial_{Ya}\} = \partial_{Cas\,(a)} - (\lambda_p - \lambda_{p-1})\partial_a. \tag{4.2.6}$$

Taking transposes of the operators in (4.2.6) (using Lemma 2.1.4, and moving up the value of p by one) we have

$$-2\,\mathrm{trace}\,\{(X, Y) \mapsto \delta_{Ya}X\} = \delta_{Cas\,(a)} - (\lambda_{p+1} - \lambda_p)\delta_a.$$

In terms of the orthonormal bases $\{X_s\}_{s=1}^n \subset \mathcal{G}$ and $\{e_i\}_{i=0}^m \subset \mathbf{R}^{m+1}$, this is rewritten as

$$2\sum_{s=1}^{n}\sum_{i=0}^{m}\langle X_s e_i, a\rangle \delta_i X_s \chi = \delta_{Cas\,(a)}\chi - (\lambda_{p+1} - \lambda_p)\delta_a\chi,\ \chi \in \mathcal{H}^p. \tag{4.2.7}$$

Recall from Section 3.5 that the homomorphism Ψ_p^k, evaluated on a full minimal immersion $f : S^m \to S_V$ of degree p and order of isotropy $k-1$, $k \geq 2$, measures how far f is from being isotropic of order k. The preservance of isotropy under $\mathcal{A}_p$ will follow from a formula below that shows how the operator $\mathcal{A}_p$ interacts with Ψ_p^k.

To express this interaction in convenient terms, we introduce the following notation. With f as above, we let

$$\Xi_p^k(f)(a_1, \ldots, a_{2k}), \quad a_1, \ldots, a_{2k} \in \mathbf{R}^{m+1},$$

be the trace of the bilinear form

$$(X, Y) \mapsto X \cdot \sum_{j=1}^{2k} \Psi_p^k(f)(a_1, \ldots, a_{j-1}, Ya_j, a_{j+1}, \ldots, a_{2k}) \tag{4.2.8}$$

on $\mathcal{G}$ (with values in $\mathcal{P}^{2(p-k)}$).

Theorem 4.2.4. *Let $f : S^m \to S_V$ be a full isotropic minimal immersion of degree p and order of isotropy $k-1$, $k \geq 2$. Then, for $a_1, \ldots, a_{2k} \in \mathbf{R}^{m+1}$,*

we have

$$\begin{aligned}
&2\lambda_p \Psi_p^k(\hat{f})(a_1,\dots,a_{2k}) \\
&\quad = (2\lambda_p - \mu^{2(p-k),2k} + 2km - 4k)\Psi_p^k(f)(a_1,\dots,a_{2k}) \\
&\quad - \sum_{j=1}^{2k} \Psi_p^k(f)(a_1,\dots,a_{j-1},\ Cas\,(a_j), a_{j+1},\dots,a_{2k}) \qquad (4.2.9) \\
&+2\Xi_p^k(f)(a_1,\dots,a_{2k}) + \lambda_{2p}\Psi_{p-1}^k(f^-)(a_1,\dots,a_{2k}),
\end{aligned}$$

where $\mu^{2(p-k),2k}$ *is given in (4.2.3).*

For $G = SO(m+1)$, *we have*

$$\Xi_p^k(f)(a_1,\dots,a_{2k}) = 2k\Psi_p^k(f)(a_1,\dots,a_{2k}), \qquad (4.2.10)$$

in particular

$$\Lambda_p^{2(p-k),2k} = 1 - \frac{\mu^{2(p-k),2k}}{2\lambda_p}. \qquad (4.2.11)$$

If $f : S^m \to S_V$ is a full isotropic minimal immersion of degree p and order of isotropy k, $k \geq 2$, then $\Psi_p^k(f) = 0$ and hence $\Xi_p^k(f) = 0$, so that (4.2.9) (along with (3.5.25)) imply that $\Psi_p^k(\hat{f}) = 0$. This is the first statement of Theorem 4.2.1. Moreover, for $G = SO(m+1)$, substituting (4.2.10) into (4.2.9), we see that $\Psi_p^k(\hat{f})$ is a linear combination of $\Psi_p^k(f)$ and $\Psi_{p-1}^k(f^-)$. The Casimir operator on $\mathbf{R}^{m+1} \cong \mathcal{H}_m^1$ is multiplication by $m = \lambda_1$ so that the contribution from the second term on the right-hand side of (4.2.9) is $-2km\Psi_p^k(f)$. By (4.2.10), the contribution from the third term is $4k\Psi_p^k(f)$. Thus the overall contributions from the first terms amount to $(2\lambda_p - \mu^{2(p-k),2k})\Psi_p^k(f)$. If, in addition, $\langle f\rangle \in V^{(2(p-k),2k,0,\dots,0)}$, then $\Psi_{p-1}^k(f^-)$ (and hence the fourth term) vanish (Corollary 3.1.5) and $\Psi_p^k(\hat{f})$ becomes a constant multiple of $\Psi_p^k(f)$. The constant then must be $\Lambda_p^{2(p-k),2k}$. (4.2.11) follows.

The remainder of this section is devoted to the proof of Theorem 4.2.4. This will also complete the proof of Theorem 4.2.1 with the exception of extending (4.2.11) to all even coordinate points (u,v) in $\triangle_1^p$, i.e. (4.2.2)-(4.2.3). This extension will be accomplished in the next section.

PROOF OF THEOREM 4.2.4. In view of (3.5.5) and (4.1.1), to obtain the stated formula for $\Psi_p^k(\langle f\rangle)$, we need to work out

$$\begin{aligned}
&\langle \partial_{a_1}\dots\partial_{a_k}\hat{f}, \partial_{a_{k+1}}\dots\partial_{a_{2k}}\hat{f}\rangle \\
&\quad = \frac{1}{\lambda_p}\sum_{s=1}^{n}\langle \partial_{a_1}\dots\partial_{a_k}X_s f, \partial_{a_{k+1}}\dots\partial_{a_{2k}}X_s f\rangle, \qquad (4.2.12)
\end{aligned}$$

where, as usual, $\{X_s\}_{s=1}^n \subset \mathcal{G}$ is an orthonormal basis. We can use Lemma 4.2.2 to switch X_s with the directional derivatives. We obtain

$$\begin{aligned}\partial_{a_1}\dots\partial_{a_k}X_sf &= \partial_{a_1}\dots\partial_{a_{k-1}}[\partial_{a_k},X_s]f+\partial_{a_1}\dots\partial_{a_{k-1}}X_s\partial_{a_k}f\\ &= \partial_{X_sa_k}\partial_{a_1}\dots\partial_{a_{k-1}}f+\partial_{a_1}\dots\partial_{a_{k-1}}X_s\partial_{a_k}f\\ &= \sum_{j=1}^{k}\partial_{X_sa_j}\partial_{a_1}\dots\widehat{\partial_{a_j}}\dots\partial_{a_k}f\\ &\qquad +X_s\partial_{a_1}\dots\partial_{a_k}f. \end{aligned}\tag{4.2.13}$$

Here and in what follows, $\widehat{\ }$ means that the corresponding factor is absent. We write the right-hand side of (4.2.13) as A_1+A_2, where

$$A_1=\sum_{j=1}^{k}\partial_{X_sa_j}\partial_{a_1}\dots\widehat{\partial_{a_j}}\dots\partial_{a_k}f$$

and

$$A_2=X_s\partial_{a_1}\dots\partial_{a_k}f,$$

and as B_1+B_2 when $a_1,\dots,a_k$ are replaced by $a_{k+1},\dots,a_{2k}$, where

$$B_1=\sum_{j=k+1}^{2k}\partial_{X_sa_j}\partial_{a_{k+1}}\dots\widehat{\partial_{a_j}}\dots\partial_{a_{2k}}f$$

and

$$B_2=X_s\partial_{a_{k+1}}\dots\partial_{a_{2k}}f.$$

With this notation, in order to compute (4.2.12), we need to work out

$$\sum_{s=1}^{n}\sum_{\alpha,\beta=1}^{2}\langle A_\alpha,B_\beta\rangle.$$

To further simplify the computations we introduce the notation

$$F(a_1,\dots,a_l)=\partial_{a_1}\dots\partial_{a_l}f\in\mathcal{H}^{p-l}.$$

By definition

$$\partial_{a_0}F(a_1,\dots,a_l)=F(a_0,a_1,\dots,a_l),\quad a_0\in\mathbf{R}^{m+1}.$$

Note first that, by (2.4.2) and harmonicity of f, we have

$$\begin{aligned}&\triangle\langle F(a_1,\dots,a_l),F(a_{l+1},\dots,a_{2l})\rangle\\ &\quad=2\sum_{i=0}^{m}\langle F(e_i,a_1,\dots,a_l),F(e_i,a_{l+1},\dots,a_{2l})\rangle,\end{aligned}\tag{4.2.14}$$

where, as usual, $\triangle$ is the Euclidean Laplacian and $\{e_i\}_{i=0}^m\subset\mathbf{R}^{m+1}$ is the standard basis. To obtain (4.2.9), we now work out each scalar product

$\langle A_\alpha, B_\beta\rangle$. By (4.2.13) and the definition of F, we have

$$\sum_{s=1}^{n}\langle A_1, B_1\rangle = \sum_{s=1}^{n}\sum_{j=1}^{k}\sum_{l=k+1}^{2k} \tag{4.2.15}$$
$$\langle F(X_s a_j, a_1, \ldots, \widehat{a_j}, \ldots, a_k), F(X_s a_l, a_{k+1}, \ldots, \widehat{a_l}, \ldots, a_{2k})\rangle.$$

Recall from our study of higher order isotropy (Section 3.5) that in the difference of the right-hand side of (4.2.15) and the analogous terms when f is replaced by f_p, the partial derivatives can be permuted (cf. (3.5.7) in Lemma 3.5.1). Hence, up to sign, we need to consider

$$\sum_{s=1}^{n}\sum_{j=1}^{k}\sum_{l=k+1}^{2k}$$
$$\langle F(X_s a_j, X_s a_l, a_1, \ldots, \widehat{a_j}, \ldots, a_k), F(a_{k+1}, \ldots, \widehat{a_l}, \ldots, a_{2k})\rangle$$
$$= \sum_{j=1}^{k}\sum_{l=k+1}^{2k}$$
$$\left\langle \sum_{s=1}^{n}\left(\partial_{X_s a_j}\partial_{X_s a_l}\right) F(a_1, \ldots, \widehat{a_j}, \ldots, a_k), F(a_{k+1}, \ldots, \widehat{a_l}, \ldots, a_{2k})\right\rangle.$$

The differential operator $\sum_{s=1}^{n}\partial_{X_s a}\partial_{X_s b}$, $a, b \in \mathbf{R}^{m+1}$, *acting on* $\mathcal{H}^{p-k+1}$ (since $F(a_1, \ldots, \widehat{a_j}, \ldots, a_k)$ is of degree $p-k+1$), can be worked out using Lemma 4.2.2 and (4.2.6), as follows

$$\begin{aligned}
\sum_{s=1}^{n}\partial_{X_s a}\partial_{X_s b} &= \sum_{s=1}^{n}[\partial_a, X_s]\partial_{X_s b} \\
&= \partial_a \sum_{s=1}^{n} X_s \partial_{X_s b} - \sum_{s=1}^{n} X_s \partial_{X_s b}\partial_a \\
&= \frac{1}{2}\partial_a\left(\partial_{Cas\,(b)} - (\lambda_{p-k+1} - \lambda_{p-k})\partial_b\right) \\
&\quad -\frac{1}{2}\left(\partial_{Cas\,(b)} - (\lambda_{p-k} - \lambda_{p-k-1})\partial_b\right)\partial_a \\
&= -\frac{1}{2}(\lambda_{p-k+1} + \lambda_{p-k-1} - 2\lambda_{p-k})\partial_a\partial_b \\
&= -\partial_a\partial_b.
\end{aligned}$$

In the next-to-last equality, we used the fact that

$$\begin{aligned}
&\lambda_{p-k+1} + \lambda_{p-k-1} - 2\lambda_{p-k} \\
&\quad = (p-k+1)((p-k+1)+m-1) + (p-k-1)((p-k-1)+m-1) \\
&\qquad -2(p-k)(p-k+m-1) \\
&\qquad = (p-k+1)^2 + (p-k-1)^2 - 2(p-k)^2 = 2.
\end{aligned}$$

Permuting the partial derivatives back, we obtain

$$\begin{aligned}\sum_{s=1}^{n}\langle A_1, B_1\rangle &= -\sum_{j=1}^{k}\sum_{l=k+1}^{2k} \\ &\quad \langle F(a_j, a_1, \ldots, \widehat{a_j}, \ldots, a_k), F(a_l, a_{k+1}, \ldots, \widehat{a_l}, \ldots, a_{2k})\rangle \\ &= -k^2\langle F(a_1, \ldots, a_k), F(a_{k+1}, \ldots, a_{2k})\rangle.\end{aligned}$$

Summarizing, and suppressing the arguments $(a_1, \ldots, a_{2k})$, the terms coming from $\langle A_1, B_1\rangle$ contribute to $2\lambda_p\Psi_p^k(\hat{f})$ the term $-2k^2\Psi_p^k(f)$.

We can now work out the terms coming from $\langle A_1, B_2\rangle$. We have

$$\begin{aligned}\sum_{s=1}^{n}\langle A_1, B_2\rangle &= \sum_{s=1}^{n}\sum_{j=1}^{k} \\ &\quad \langle \partial_{X_s a_j} F(a_1, \ldots, \widehat{a_j}, \ldots, a_k), X_s F(a_{k+1}, \ldots, a_{2k})\rangle \\ &= \sum_{s=1}^{n}\sum_{j=1}^{k} X_s\langle \partial_{X_s a_j} F(a_1, \ldots, \widehat{a_j}, \ldots, a_k), \\ &\qquad F(a_{k+1}, \ldots, a_{2k})\rangle \\ &\qquad -\sum_{j=1}^{k}\left\langle \sum_{s=1}^{n} X_s \partial_{X_s a_j} F(a_1, \ldots, \widehat{a_j}, \ldots, a_k), \right. \\ &\qquad \left. F(a_{k+1}, \ldots, a_{2k})\right\rangle.\end{aligned}$$

By (4.2.8), the first sum on the right-hand side along with the analogous sum from $\sum_{s=1}^{n}\langle A_2, B_1\rangle$ contribute $2\Xi_p^k(f)$ to $2\lambda_p\Psi_p^k(\hat{f})$. By (4.2.6), the second sum on the right-hand side is rewritten as

$$\begin{aligned}&-\frac{1}{2}\sum_{j=1}^{k}\langle F(a_1, \ldots, a_{j-1}, \mathit{Cas}\,(a_j), a_{j+1}, \ldots, a_k), F(a_{k+1}, \ldots, a_{2k})\rangle \\ &\quad +\frac{k}{2}(\lambda_{p-k+1} - \lambda_{p-k})\langle F(a_1, \ldots, a_k), F(a_{k+1}, \ldots, a_{2k})\rangle.\end{aligned}$$

This contributes to $2\lambda_p\Psi_p^k(\hat{f})$ the term

$$\begin{aligned}&-\sum_{j=1}^{k}\Psi_p^k(f)(a_1, \ldots, a_{j-1}, \mathit{Cas}\,(a_j), a_{j+1}, \ldots, a_k, a_{k+1}, \ldots, a_{2k}) \\ &\quad +k(\lambda_{p-k+1} - \lambda_{p-k})\Psi_p^k(f)(a_1, \ldots, a_{2k}).\end{aligned}$$

The unaccounted terms in $\sum_{i=1}^{s}\langle A_1, B_2\rangle$ can be treated analogously.

Finally, for the terms coming from $\langle A_2, B_2\rangle$, we compute

$$2\sum_{s=1}^{n}\langle A_2, B_2\rangle = 2\sum_{s=1}^{n}\langle X_s F(a_1, \ldots, a_k), X_s F(a_{k+1}, \ldots, a_{2k})\rangle$$

$$
\begin{aligned}
&= -\triangle^{S^m}\langle F(a_1,\dots,a_k), F(a_{k+1},\dots,a_{2k})\rangle \\
&\quad +\langle \triangle^{S^m} F(a_1,\dots,a_k), F(a_{k+1},\dots,a_{2k})\rangle \\
&\quad +\langle F(a_1,\dots,a_k), \triangle^{S^m} F(a_{k+1},\dots,a_{2k})\rangle \\
&= \triangle\langle F(a_1,\dots,a_k), F(a_{k+1},\dots,a_{2k})\rangle \\
&\quad +(2\lambda_{p-k} - \lambda_{2(p-k)})\langle F(a_1,\dots,a_k), F(a_{k+1},\dots,a_{2k})\rangle
\end{aligned}
$$

(where we doubled for convenience). In this computation we used the fact that the components of f are spherical harmonics on S^m of order p, we also used the connecting formula between the Euclidean and spherical Laplacians (Section 2.1), and, finally, that $\triangle^{S^m} = -\sum_{s=1}^{n} X_s^2$. Summarizing, we see that the terms under consideration contribute to $2\lambda_p \Psi_p^k(\hat{f})$ the terms $\lambda_{2p}\Psi_{p-1}^k(f^-)$ (by (3.5.25)) and $(2\lambda_{p-k} - \lambda_{2(p-k)})\Psi_p^k(f)$.

Putting all the contributions together, we arrive at (4.2.9).

We now turn to the proof of (4.2.10) assuming $G = SO(m+1)$. We fix the usual orthonormal basis $\{E_{rs}\}_{0\le r<s\le m} \subset so(m+1)$, where $E_{rs} = x_s\partial_r - x_r\partial_s$, $r, s = 0,\dots,m$. We need to work out the trace of the bilinear form (4.2.8). To do this, we compute

$$
\begin{aligned}
&\sum_{0\le r<s\le m}\sum_{j=1}^{k} E_{rs}\langle \partial_{E_{rs}a_j} F(a_1,\dots,\widehat{a_j},\dots,a_k), F(a_{k+1},\dots,a_{2k})\rangle \\
&= \frac{1}{2}\sum_{r,s=0}^{m}\sum_{j=1}^{k}(x_s\partial_r - x_r\partial_s) \\
&\quad \times\langle (a_{j,s}\partial_r - a_{j,r}\partial_s)F(a_1,\dots,\widehat{a_j},\dots,a_k), F(a_{k+1},\dots,a_{2k})\rangle \\
&= \sum_{r=0}^{m}\sum_{j=1}^{k}\langle a_j, x\rangle\partial_r\langle \partial_r F(a_1,\dots,\widehat{a_j},\dots,a_k), F(a_{k+1},\dots,a_{2k})\rangle \\
&\quad -\sum_{r=0}^{m}\sum_{j=1}^{k} x_r\partial_{a_j}\langle \partial_r F(a_1,\dots,\widehat{a_j},\dots,a_k), F(a_{k+1},\dots,a_{2k})\rangle \\
&= \frac{1}{2}\sum_{j=1}^{k}\langle a_j, x\rangle\triangle\langle F(a_1,\dots,\widehat{a_j},\dots,a_k), F(a_{k+1},\dots,a_{2k})\rangle \\
&\quad +\sum_{r=0}^{m}\sum_{j=1}^{k} a_{j,r}\langle F(e_r, a_1,\dots,\widehat{a_j},\dots,a_k), F(a_{k+1},\dots,a_{2k})\rangle \\
&\quad -\sum_{j=1}^{k}\partial_{a_j}\left\langle \sum_{r=0}^{m} x_r\partial_r F(a_1,\dots,\widehat{a_j},\dots,a_k), F(a_{k+1},\dots,a_{2k})\right\rangle \\
&= \frac{1}{2}\sum_{j=1}^{k}\langle a_j, x\rangle\triangle\langle F(a_1,\dots,\widehat{a_j},\dots,a_k), F(a_{k+1},\dots,a_{2k})\rangle \\
&\quad +k\langle F(a_1,\dots,a_k), F(a_{k+1},\dots,a_{2k})\rangle
\end{aligned}
$$

$$-(p-k+1)\sum_{j=1}^{k}\partial_{a_j}\langle F(a_1,\ldots,\widehat{a_j},\ldots,a_k),F(a_{k+1},\ldots,a_{2k})\rangle.$$

The first and last sums in the resulting expression contribute zero by (3.5.6) since $\Psi_p^{k-1}(f)=0$, and the middle term contributes $k\Psi_p^k(f)(a_1,\ldots,a_{2k})$. Moving up the value of the vectorial index by k, (4.2.10) follows. Theorem 4.2.4 is proved.

4.3 Infinitesimal Rotations and Degree-Raising

The degree-raising operator (Section 2.4) and the operator of infinitesimal rotations interact in a particularly beautiful way. In this section, we derive a formula relating these two operators. As a byproduct, this will enable us to extend the eigenvalue formula (4.2.11) to (4.2.2)-(4.2.3), and thereby to finish the proof of Theorem 4.2.1.

Recall the triangle $\triangle_1^p$ in (3.1.9) with vertices $(2,2)$, (p,p), $(2(p-1),2)$, whose even coordinate points give the nonzero components of the highest weights of the irreducible representations that occur in $\mathcal{E}^p$. For $p\le q$, we have $\triangle_1^p\subset\triangle_1^q$, so that $\mathcal{E}^p$ is an $SO(m+1)$-submodule of $\mathcal{E}^q$. Recall from Theorem 2.6.1 that a specific linear imbedding is given by $(\Phi^+)^{q-p}:\mathcal{E}^p\to\mathcal{E}^q$, where Φ^+ is the degree-raising operator introduced in Section 2.4. The following result (formulated for $q=p+1$) shows that the eigenvalues of $\mathcal{A}_p$ on $\mathcal{E}^p$ determine the eigenvalues of $\mathcal{A}_q$ on $\mathcal{E}^p\subset\mathcal{E}^q$.

Theorem 4.3.1. *Let $G\subset SO(m+1)$ be a closed subgroup acting transitively on S^m, and $\mathcal{A}_p$ the operator of infinitesimal rotations associated to G. Then, for $C\in S^2(\mathcal{H}^p)$, we have*

$$\lambda_p(\Phi_p^+(C)-\Phi_p^+(\mathcal{A}_p(C)))=\lambda_{p+1}(\Phi_p^+(C)-\mathcal{A}_{p+1}(\Phi_p^+(C))). \tag{4.3.1}$$

Recall from Theorem 4.2.1 that $\Lambda_p^{u,v}$ denotes the eigenvalue of $\mathcal{A}_p$ on the irreducible component $V^{(u,v,0,\ldots,0)}$ of $\mathcal{E}^p$ ($(u,v)\in\triangle_1^p$, u,v even). Restricting C to $V^{(u,v,0,\ldots,0)}$ in Theorem 4.3.1 and using induction with respect to p, we immediately obtain the following:

Corollary 4.3.2. *We have*

$$\lambda_p(1-\Lambda_p^{u,v})=\lambda_q(1-\Lambda_q^{u,v}). \tag{4.3.2}$$

Before doing the proof of Theorem 4.3.1, we note that (4.3.2) combined with (4.2.11) now give all eigenvalues of $\mathcal{A}_p$ on $\mathcal{E}^p$.

Indeed, for $(u,v) \in \triangle_1^p$, with u, v even, (4.2.11) is rewritten as

$$\Lambda^{u,v}_{\frac{u+v}{2}} = 1 - \frac{\mu^{u,v}}{2\lambda_{\frac{u+v}{2}}}.$$

(4.3.2) with $q = \frac{u+v}{2}$ now gives us

$$\Lambda_p^{u,v} = 1 - \frac{\lambda_q}{\lambda_p}(1 - \Lambda_q^{u,v}) = 1 - \frac{\lambda_q}{\lambda_p}\frac{\mu^{u,v}}{2\lambda_q} = 1 - \frac{\mu^{u,v}}{2\lambda_p}.$$

This is (4.2.2). To finish the proof of Theorem 4.2.1 it now remains for us to work out (4.3.1).

PROOF OF THEOREM 4.3.1. Let $C \in S^2(\mathcal{H}^p)$. We work out

$$\lambda_{p+1}\mathcal{A}_{p+1}(\Phi_p^+(C)) - \lambda_p \Phi_p^+(\mathcal{A}_p(C)). \tag{4.3.3}$$

As usual, we fix an orthonormal basis $\{X_s\}_{s=1}^n \subset \mathcal{G}$. Using the definition of the degree-raising-operator Φ_p^+ (Section 2.4), (2.1.20) and (2.1.22), for $\chi \in \mathcal{H}^{p+1}$, we have

$$\begin{aligned}
\lambda_{p+1}\mathcal{A}_{p+1}(\Phi_p^+(C))(\chi) &= \lambda_{p+1}\alpha_{p+1}^\top(\Phi_p^+(C)\otimes I)\alpha_{p+1}(\chi) \\
&= -\sum_{s=1}^n X_s(\Phi_p^+(C)X_s\chi) \\
&= -c_p^+ \sum_{s=1}^n \sum_{i=0}^m X_s(\iota_+^\top(C\otimes I)\partial_i(X_s\chi)\otimes y_i) \\
&= -\frac{1}{\sqrt{p+1}} \sum_{s=1}^n \sum_{i=0}^m H(X_s(x_i C(\partial_i(X_s\chi)))).
\end{aligned}$$

In the last equality, we used the fact that infinitesimal isometries commute with H. On the other hand, we have

$$\begin{aligned}
\lambda_p \Phi_p^+(\mathcal{A}_p(C))(\chi) &= \lambda_p \iota_+^\top(\mathcal{A}_p(C)\otimes I)\iota_+(\chi) \\
&= \lambda_p c_p^+ \sum_{i=0}^m \iota_+^\top(\mathcal{A}_p(C)\otimes I)(\partial_i\chi \otimes y_i) \\
&= \frac{\lambda_p}{\sqrt{p+1}} \sum_{i=0}^m H(x_i \mathcal{A}_p(C)\partial_i\chi) \\
&= -\frac{1}{\sqrt{p+1}} \sum_{s=1}^n \sum_{i=0}^m H(x_i X_s C X_s \partial_i\chi).
\end{aligned}$$

The difference (4.3.3) of these two terms is

$$-\frac{1}{\sqrt{p+1}} \sum_{s=1}^n \sum_{i=0}^m \big(H(X_s(x_i C(\partial_i X_s\chi))) - H(x_i X_s C(X_s\partial_i\chi))\big).$$

Performing differentiation in the first term, the entire sum (suppressing the coefficient $-1/\sqrt{p+1}$) can be rewritten as

$$\sum_{s=1}^{n}\sum_{i=0}^{m} H(X_s(x_i)C(\partial_i X_s\chi)) + \sum_{s=1}^{n}\sum_{i=0}^{m} H(x_i X_s C[\partial_i, X_s]\chi). \tag{4.3.4}$$

We write this as $I + II$ and work out each term separately. Using the standard basis $\{e_i\}_{i=0}^{m} \subset \mathbf{R}^{m+1}$, we have

$$\begin{aligned} I &= \sum_{s=1}^{n}\sum_{i=0}^{m} H(\langle X_s x, e_i\rangle C(\partial_i X_s\chi)) \\ &= \sum_{s=1}^{n}\sum_{i,r=0}^{m} H(x_r\langle X_s e_r, e_i\rangle C(\partial_i X_s\chi)) \\ &= \sum_{s=1}^{n}\sum_{r=0}^{m} \delta_r C\partial_{X_s e_r} X_s\chi. \end{aligned}$$

By Lemmas 4.2.2-4.2.3, the second-order operator

$$\begin{aligned} \partial_{X_s e_r} X_s &= [\partial_{X_s e_r}, X_s] + X_s\partial_{X_s e_r} \\ &= \partial_{X_s^2 e_r} + \frac{1}{2}(-\partial_{X_s^2 e_r} + \partial_r (X_s)^2 - (X_s)^2\partial_r) \\ &= \frac{1}{2}(\partial_{X_s^2 e_r} + \partial_r (X_s)^2 - (X_s)^2\partial_r) \end{aligned}$$

We now sum up with respect to $s = 1, \ldots, n$. Using the fact that $\sum_{s=1}^{n} X_s^2 = -\triangle^{S^m} = -\lambda_p I$ on $\mathcal{H}^p$, we obtain

$$I = -\frac{1}{2}\sum_{r=0}^{m} \delta_r C\partial_{Cas\,(e_r)}\chi - \frac{1}{2}(\lambda_{p+1} - \lambda_p)\sum_{r=0}^{m} \delta_r C\partial_r\chi, \tag{4.3.5}$$

where $Cas\,(a) = -\sum_{s=1}^{n} X_s^2 a$, is the Casimir operator of a on $\mathbf{R}^{m+1}$.

By Lemma 4.2.2, the second sum in (4.3.4) is rewritten as

$$II = \sum_{s=1}^{n}\sum_{i=0}^{m} \delta_i X_s C[\partial_i, X_s]\chi = \sum_{s=1}^{n}\sum_{i=0}^{m} \langle X_s e_i, e_r\rangle \delta_i X_s C\partial_r\chi.$$

This is (4.2.7) with $a = e_r$ and $C\partial_r\chi$ substituted for χ. We obtain

$$II = \frac{1}{2}\sum_{r=0}^{m} \delta_{Cas\,(e_r)} C\partial_r\chi - \frac{1}{2}(\lambda_{p+1} - \lambda_p)\sum_{r=0}^{m} \delta_r C\partial_r\chi.$$

Combining this with (4.3.5), we arrive at

$$I + II = -(\lambda_{p+1} - \lambda_p)\sum_{r=0}^{m} \delta_r C\partial_r\chi.$$

(Notice that the first terms in I and II cancel by the symmetry of the Casimir operator.) We finally obtain that the difference in (4.3.3) is equal

to

$$\frac{\lambda_{p+1}-\lambda_p}{\sqrt{p+1}}\sum_{r=0}^{m}\delta_r C\partial_r\chi=(\lambda_{p+1}-\lambda_p)\Phi_p^+(C)(\chi),$$

where we used the definition of the degree-raising operator. Theorem 4.3.1 follows.

4.4 Lower Bounds for the Range Dimension, Part I

We have now come to the main point of this chapter, the use of the operator of infinitesimal rotations to give lower bounds for the range of eigenmaps and spherical minimal immersions.

For the simplest example, let $G\subset SO(m+1)$ be a (closed) semisimple subgroup acting transitively on S^m, and $f: S^m\to S_V$ a full p-eigenmap. Assume that $\langle f\rangle$ is contained in an $\mathcal{A}_p$-invariant irreducible G-submodule $\mathcal{V}\subset\mathcal{E}^p$ orthogonal to $(\mathcal{E}^p)^G$. The operator $\mathcal{A}_p$ acts on $\mathcal{V}$ by a single eigenvalue Λ. Theorem 4.1.1 asserts that Λ is contained in $(-1,1)$. First, assume that Λ is nonnegative. In this case $\mathcal{A}_p(\langle f\rangle)=\Lambda\langle f\rangle$ must be contained in the interior of $\mathcal{L}^p$. The p-eigenmap representing $\mathcal{A}_p(\langle f\rangle)$ is $\hat{f}$ defined by (4.1.1), and its range is contained in $V\otimes\mathcal{G}^*$. On the other hand, we just noted that $\langle\hat{f}\rangle=\mathcal{A}_p(\langle f\rangle)$ is in the interior of $\mathcal{L}^p$, in particular, the space of components of $\hat{f}$ is maximal: $V_{\hat{f}}=\mathcal{H}^p$. Comparing dimensions, we thus have

$$\dim\mathcal{H}^p\le\dim(V\otimes\mathcal{G}^*)=\dim V\cdot\dim G.$$

We obtain the lower estimate

$$\dim V\ge\frac{\dim\mathcal{H}^p}{\dim G}.$$

Second, if Λ is negative then either $\langle\hat{f}\rangle$ is in the interior of $\mathcal{L}^p$, in which case the estimate above applies, or $\langle\hat{f}\rangle$ is the opposite of $\langle f\rangle$ on the boundary of $\mathcal{L}^p$. In the latter case, the segment joining $\langle f\rangle$ and $\langle\hat{f}\rangle$ contains the origin, an interior point of $\mathcal{L}^p$. By Theorem 2.3.5, we have $V_f+V_{\hat{f}}=\mathcal{H}^p$. We thus have

$$\begin{aligned}\dim\mathcal{H}^p&\le\dim V_f+\dim V_{\hat{f}}\\&\le\dim V+\dim V\cdot\dim\mathcal{G}^*\\&=\dim V(1+\dim G).\end{aligned}$$

We obtain the slightly weaker lower estimate

$$\dim V\ge\frac{\dim\mathcal{H}^p}{\dim G+1}.$$

(Note also that in this case $(\mathcal{A}_p)^2(\langle f\rangle)$ is in the interior of $\mathcal{L}^p$.) We proved the following:

Theorem 4.4.1. *Let G be semisimple and transitive on S^m, and assume that the congruence class of a full p-eigenmap $f : S^m \to S_V$ is contained in an $\mathcal{A}_p$-invariant irreducible G-submodule $\mathcal{V}$ of $\mathcal{E}^p$ orthogonal to $(\mathcal{E}^p_m)^G$. Let $\Lambda \in (-1,1)$ denote the eigenvalue of $\mathcal{A}_p$ on $\mathcal{V}$. If $\Lambda \geq 0$ then*

$$\dim V \geq \frac{\dim \mathcal{H}^p}{\dim G}. \tag{4.4.1}$$

If $\Lambda < 0$ then

$$\dim V \geq \frac{\dim \mathcal{H}^p}{\dim G + 1}. \tag{4.4.2}$$

The simplest case where we can apply Theorem 4.4.1 is when $G = SO(m+1)$, $m \geq 4$, since there are no nonzero $SO(m+1)$ fixed points in $\mathcal{E}^p$, and the eigenvalue $\Lambda_p^{u,v}$ of $\mathcal{A}_p$ on the irreducible component $V^{(u,v,0,\ldots,0)} \subset \mathcal{E}^p$ is known from Theorem 4.2.1. We will first study this particular case, and then allow $\langle f\rangle$ to be contained in a *reducible* $SO(m+1)$-submodule in $\mathcal{E}^p$.

The case of $SU(2)$-equivariant p-eigenmaps will be treated in detail in the next section.

Regarding u and v as continuous variables, the equation $\Lambda_p^{u,v} = 0$ defines the circle

$$\left(u + \frac{m-1}{2}\right)^2 + \left(v + \frac{m-3}{2}\right)^2 = 2\lambda_p + \left(\frac{m-1}{2}\right)^2 + \left(\frac{m-3}{2}\right)^2. \tag{4.4.3}$$

The eigenvalue $\Lambda_p^{u,v}$ is positive iff the lattice point $(u,v) \in \triangle_1^p$ (with even coordinates) is inside this circle. Figures 27-28 depict the situations for $m = 4$, $p = 40$, and $m = 1000$, $p = 40$.

Theorem 4.4.1 immediately gives us the following:

Corollary 4.4.2. *Let $f : S^m \to S_V$, $m \geq 4$, be a full minimal immersion of degree p. Assume that $\langle f\rangle$ is contained in $V^{(u,v,0,\ldots,0)} \subset \mathcal{F}^p$. If $\Lambda_p^{u,v} \geq 0$, i.e.*

$$u^2 + v^2 + u(m-1) + v(m-3) \leq 2p(p+m-1),$$

then we have

$$\dim V \geq \frac{\dim \mathcal{H}^p}{\dim so(m+1)} = \frac{2(2p+m-1)(p+m-2)!}{p!(m+1)!}. \tag{4.4.4}$$

If $\Lambda_p^{u,v} < 0$, i.e.

$$u^2 + v^2 + u(m-1) + v(m-3) > 2p(p+m-1),$$

then, we have

$$\dim V \geq \frac{\dim \mathcal{H}^p}{\dim so(m+1)+1} = \frac{2(2p+m-1)(p+m-2)!}{p!(m-1)!(m(m+1)+2)}. \tag{4.4.5}$$

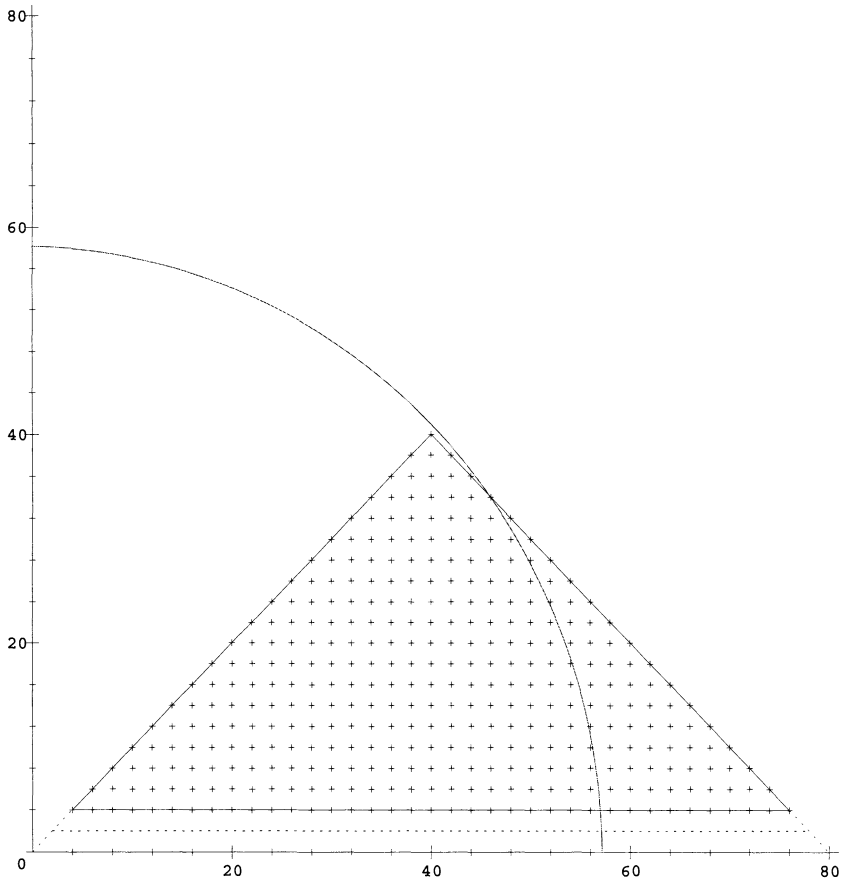


Figure 27.

Remark 1. It is interesting to compare the lower estimate for the range in the remark after Corollary 3.1.5 with (4.4.1)-(4.4.2). For example, for $m \geq 3$ and $p \geq 4$:

$$\frac{\dim \mathcal{H}^p}{\dim so(m+1)} < \frac{\dim \mathcal{H}^{p-1}}{\dim \mathcal{H}^1}$$

(see Problem 4.7).

Remark 2. Some eigenvalues of $\mathcal{A}_p$ may be zero on $\mathcal{F}^p = \mathcal{F}^p_m$. In fact, a simple computation in the use of (4.4.3) shows that $\Lambda^{u,v}_p = 0$ ($(u,v) \in \triangle^p_2$, u, v even) *for all* $m \geq 4$, iff

$$u = l(l+2), \quad v = l^2, \text{ and } p = l(l+1) \tag{4.4.6}$$

for some even l. Since, by the DoCarmo-Wallach parametrization, the origin corresponds to the standard minimal immersion, we see that for a full

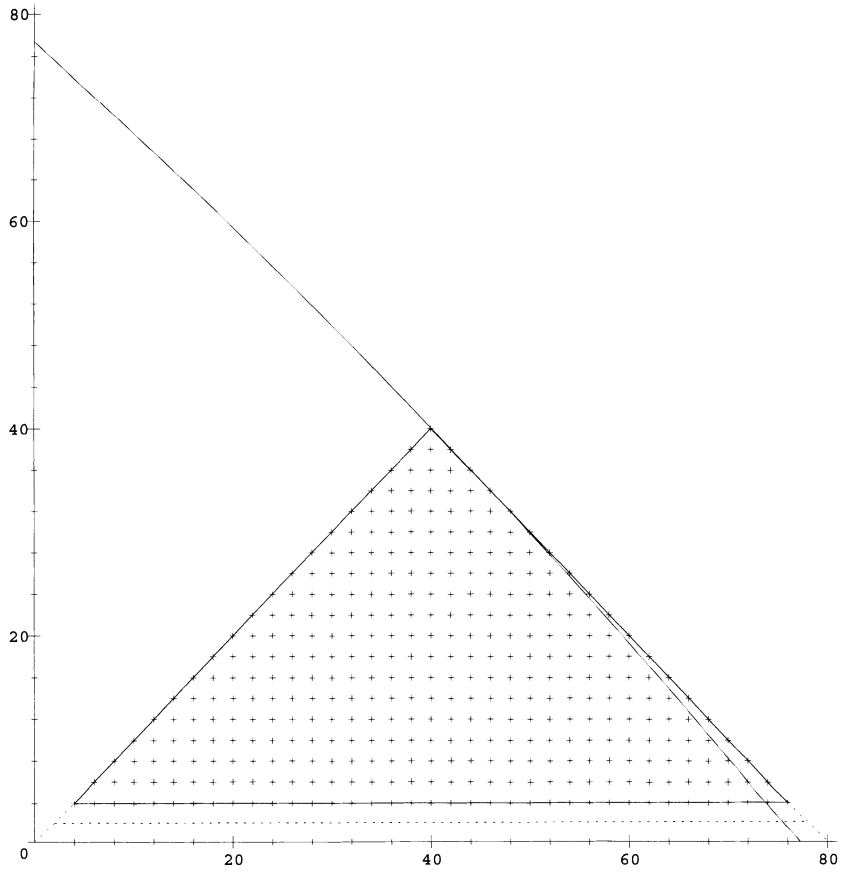


Figure 28.

minimal immersion $f : S^m \to S_V$, $m \geq 4$, of degree $l(l+1)$, l even, with $\langle f \rangle \in V^{(l(l+2),l^2,0,\ldots,0)}$, the minimal immersion $\hat{f} : S^m \to S_{V \otimes so(m+1)^*}$ (made full) is *standard*, i.e. congruent to $f_{l(l+1)}$.

All the points in (4.4.6) correspond to the northeast edge of the triangle $\triangle_2^p$ since $u + v = l(l+2) + l^2 = 2l(l+1) = 2p$.

$\Lambda_p^{u,v}$ may vanish *for some* $m \geq 4$ for (u, v) inside $\triangle_2^p$, for example, for $m = 23$, we have $\Lambda_{12}^{18,4} = 0$.

The circle (4.4.3) intersects the line $u+v = 2p$ (containing the northeast side of the triangle $\triangle_1^p$) at two points

$$\left(p - \frac{1}{2} \mp \sqrt{p + \frac{1}{4}}, p + \frac{1}{2} \pm \sqrt{p + \frac{1}{4}}\right),$$

where the lower sign gives the intersection point on the northeast side of $\triangle_1^p$. Thus, the northern vertex (p,p) of $\triangle_1^p$ is inside the circle. Equivalently, when p is even, $\Lambda_p^{p,p}$ is positive, for all p.

Remark 1. If we impose isotropy of order $k = p/2 - 1$, p even, on our full minimal immersion $f : S^m \to S_V$ of degree p, then $\langle f \rangle$ lies in the irreducible component $\mathcal{F}^{p,p/2-1} \cong V^{(p,p,0,\ldots,0)}$ corresponding to the northern vertex (p,p) of $\triangle_1^p$ (Theorem 3.5.5). Thus, Corollary 4.4.2 immediately implies that, for $m \geq 4$ and p even, (4.4.4) holds for an isotropic minimal immersion $f : S^m \to S_V$ of degree p and order of isotropy $p/2 - 1$.

The condition of isotropy is redundant for $p = 4$. Thus, for a quartic minimal immersion $f : S^m \to S_V$, $m \geq 4$, we have

$$\dim V \geq \frac{(m+2)(m+7)}{12}.$$

Although quadratic in m, this lower estimate is weaker than (3.3.8).

Theorem 4.4.1 automatically extends to the case when $\langle f \rangle$ lies in the convex hull of slices of $\mathcal{M}^p$ by irreducible components, since the contraction argument applies. The problem of finding suitable lower bounds on the range of spherical minimal immersions thus amounts to a study of how far the moduli $\mathcal{M}^p$ is from this convex hull. As we will see, $\mathcal{M}^p$ is, in general, very far from the convex hull of its irreducible slices, i.e. $\mathcal{M}^p$ "bulges out" from this convex hull. Notice that we already met this phenomenon in Section 3.1. In fact, according to the remark after Corollary 3.1.5, for $m \geq 3$ and p large, the slice $\mathcal{L}^p \cap \sum_{j=0}^{[p/2]} V^{(2p-2j,2j,0,\ldots,0)}$ does not contain any extremal points of $\mathcal{L}^p$, so that $\mathcal{L}^p$ bulges out from the convex hull of this slice and the orthogonal complement $\mathcal{L}^p \cap \mathcal{E}^{p-1}$. We will study this bulging out phenomenon for $m = 3$ in detail in the next section.

For, the remainder of this section, we will study the situation when $\mathcal{V}$ in Theorem 4.4.1 is reducible, and so $\mathcal{A}_p$ may well have many distinct eigenvalues on $\mathcal{V}$. The estimates in Theorem 4.4.1 were derived from the fact that either $\mathcal{A}_p$ ($\Lambda \geq 0$) or $(\mathcal{A}_p)^2$ ($\Lambda < 0$) carried $\langle f \rangle$ into the interior of $\mathcal{L}^p$. The following example, due to Gauchman, shows that an arbitrary large power of a linear contraction $\mathcal{A}$ may map boundary points of an $\mathcal{A}$-invariant compact convex body $\mathcal{L}$ to boundary points.

Example 4.4.3. For $n \in \mathbf{N}$, let $\mathcal{L} \subset \mathbf{R}^2$ be the convex polygon defined by its vertices: $(1, \pm 1)$, $(1/3^l, \pm 1/2^l)$, $l = 1, \ldots, d$, and $(-1/3^d, 0)$. Let $\mathcal{A}$ be the linear contraction whose matrix is diagonal with diagonal entries $1/3$ and $1/2$. (Figure 29 depicts the polygon for $d = 3$.) Clearly, $\mathcal{A}, \mathcal{A}^2, \ldots, \mathcal{A}^d$ map the vertex $(1,1)$ to boundary points (actually vertices) of $\mathcal{L}$. Note that $\mathcal{A}^{d+1}$ maps the entire polygon into its interior.

For simplicity, we will assume from now on that G is semisimple. By Theorem 4.1.1, $\mathcal{A}_p$ is the identity map on $(\mathcal{E}^p)^G$ and a contraction on the orthogonal complement of $(\mathcal{E}^p)^G$ in $\mathcal{E}^p$.

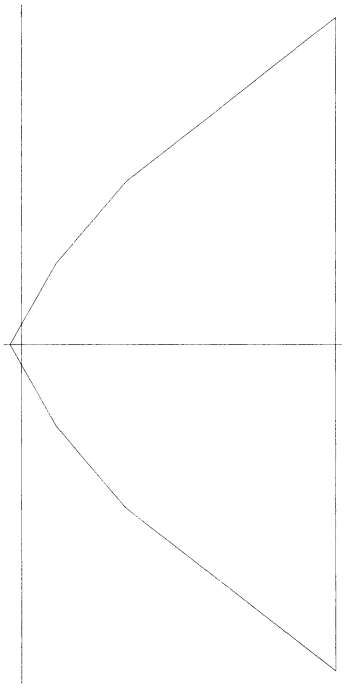


Figure 29.

Given a full p-eigenmap $f : S^m \to S_V$, we define the *critical exponent* d_f of f as follows. If $C = \langle f \rangle$ is in the interior of $\mathcal{L}^p$ then $d_f = 0$. If C is on the boundary of $\mathcal{L}^p$ then we consider the sequence $C, \mathcal{A}_p(C), (\mathcal{A}_p)^2(C), \ldots$ If all these points are on the boundary of $\mathcal{L}^p$ then $d_f = \infty$. Otherwise, d_f is the least integer $d \geq 0$ such that $(\mathcal{A}_p)^d(C)$ is in the interior of $\mathcal{L}^p$.

By definition, if C is orthogonal to $(\mathcal{E}^p)^G$ then the critical exponent d_f is finite.

Remark. If $\langle f \rangle$ is contained in an $\mathcal{A}_p$-invariant irreducible component $\mathcal{V} \subset \mathcal{E}^p$, then by the argument before Theorem 4.4.1, we have $d_f \leq 2$.

Let $f : S^m \to S_V$ be a full p-eigenmap, and assume that $2 \leq d_f < \infty$. The range of $\hat{f}^{d_f}$ ($\hat{}$ applied to f d_f times) is $V \otimes \mathcal{U}^{d_f}(\mathcal{G})^*$, where $\mathcal{U}(\mathcal{G})$ is the universal enveloping algebra of $\mathcal{G}$, and $\mathcal{U}^d(\mathcal{G})$ is the linear subspace of elements of degree $\leq d$. This follows because the components of $\hat{f}^d$ are obtained by applying monomials of degree $\leq d$ in infinitesimal isometries as variables to the components of f.

On the other hand, by definition, $\langle \hat{f}^{d_f} \rangle$ is an interior point of $\mathcal{L}^p$, so that the components of $\hat{f}^{d_f}$ span $\mathcal{H}^p$. We obtain

$$\dim \mathcal{H}^p \leq \dim V \cdot \dim(\mathcal{U}^{d_f}(\mathcal{G})^*).$$

Rewriting this, we arrive at the lower estimate

$$\dim V \geq \frac{\dim \mathcal{H}^p}{\dim(\mathcal{U}^{d_f}(\mathcal{G})^*)} = \frac{\dim \mathcal{H}^p}{\binom{\dim G + d_f}{d_f}}. \tag{4.4.7}$$

Theorem 4.4.4. *Let G be semisimple and $f : S^m \to S_V$ a full p-eigenmap. Assume that $\langle f \rangle$ is contained in an $\mathcal{A}_p$-invariant G-submodule $\mathcal{V}$ of $\mathcal{E}^p$, and that $\mathcal{V}$ is orthogonal to $(\mathcal{E}^p)^G$. Let d be the number of distinct eigenvalues of $\mathcal{A}_p$ on $\mathcal{V}$, and assume that $d \geq 2$. Then $d_f \leq d$, and we have*

$$\dim V \geq \frac{\dim \mathcal{H}^p}{\dim \mathcal{U}^d(\mathcal{G})} = \frac{\dim \mathcal{H}^p}{\binom{\dim G + d}{d}}. \tag{4.4.8}$$

PROOF. Let $\lambda_1, \ldots, \lambda_d$ denote the distinct eigenvalues of $\mathcal{A}_p|_{\mathcal{V}}$. Since $\mathcal{V}$ is orthogonal to $(\mathcal{E}^p)^G$, by Theorem 4.1.1, $\lambda_j \in (-1, 1)$, $j = 1, \ldots, d$. We may assume that f is of boundary type: $C = \langle f \rangle \in \mathcal{V} \cap \partial\mathcal{L}^p$. Let $C^l = (\mathcal{A}_p)^l(C)$, $l = 0, \ldots, d$; $C^0 = C$. We may also assume that $C^d \in \partial\mathcal{L}^p$ since otherwise $d_f \leq d$ and (4.4.8) applies.

We claim that the simplex with vertices $C^1, \ldots, C^d$ is contained in $\partial\mathcal{L}^p$.

By the definition of $\hat{f}$, the space of components $V_{\hat{f}}$ is spanned by the components of $X(f)$, $X \in \mathcal{G}$. Similarly, $V_{\hat{f}^2}$ is spanned by the components of $XY(f)$, $X, Y \in \mathcal{G}$. Since G is semisimple, $G = [G, G]$, and hence $V_{\hat{f}} \subset V_{\hat{f}^2}$. Thus, by Theorem 2.3.5, the segment joining C^1 and C^2 is on the boundary of $\mathcal{L}^p$ provided that $C^2 \in \partial\mathcal{L}^p$. Iterating $\hat{}$, we have

$$V_{\hat{f}} \subset V_{\hat{f}^2} \subset \ldots \subset V_{\hat{f}^d}. \tag{4.4.9}$$

Since $\hat{f}^l$ corresponds to C^l, $l = 1, \ldots, d$, by Theorem 2.3.5, $V_{\hat{f}^d}$ contains the spaces of components of *all* eigenmaps that correspond to points in the simplex with vertices $C^1, \ldots, C^d$. The claim follows.

Next we claim that the simplex with vertices $C = C^0, C^1, \ldots, C^d$ is not contained in the boundary of $\mathcal{L}^p$, i.e., it intersects the interior of $\mathcal{L}^p$. This will clearly imply that $d_f \leq d$. To prove this, we first notice that $C^0, \ldots, C^d$ are linearly dependent. More precisely, we have

$$\sum_{l=0}^{d} (-1)^l s_l(\lambda_1, \ldots, \lambda_d) C^{d-l} = 0, \tag{4.4.10}$$

where s_l is the l-th elementary symmetric polynomial in d variables. To show this, we write $C = \sum_{j=1}^{d} X_j$, where X_j is an eigenvector of $\mathcal{A}_p$ with eigenvalue λ_j. Applying $(\mathcal{A}_p)^{d-l}$ to both sides of this expansion, we obtain

$C^{d-l} = \sum_{j=1}^{d} \lambda_j^{d-l} X_j$. Now (4.4.10) follows from the identity

$$\sum_{l=0}^{d} (-1)^l s_l(\lambda_1, \ldots, \lambda_d) \lambda_j^{d-l} = (x - \lambda_1) \ldots (x - \lambda_d)|_{x=\lambda_j} = 0.$$

Now we rearrange (4.4.10) into convex linear combinations. To simplify the notations, we set

$$a_l = (-1)^l s_l(\lambda_1, \ldots, \lambda_d),$$

and denote

$$I^+ = \{l \,|\, a_l > 0\}, \quad I^- = \{l \,|\, a_l < 0\}$$

and

$$S^+ = \sum_{l \in I^+} a_l, \quad S^- = \sum_{l \in I^-} (-a_l).$$

By Theorem 4.4.1, we have

$$S^+ - S^- = \sum_{l=0}^{d} a_l = (1 - \lambda_1) \ldots (1 - \lambda_d) > 0. \tag{4.4.11}$$

Clearly, $0 \in I^+$. We may assume that I^- is nonempty, since otherwise our claim follows. (Indeed, if I^- is empty then, by (4.4.10), we have

$$\sum_{l \in I^+} \frac{a_l}{S^+} C^{d-l} = 0.$$

This equality shows that a convex linear combination of the vertices C^{d-l}, $l \in I^+$, gives the origin, an interior point of $\mathcal{L}^p$.)

Since $I^{\pm} \neq \emptyset$, we have $S^{\pm} > 0$. We rewrite (4.4.10) as

$$\frac{S^+}{S^-} \sum_{l \in I^+} \frac{a_l}{S^+} C^{d-l} = \sum_{l \in I^-} \frac{(-a_l)}{S^-} C^{d-l}.$$

The right-hand side of this equality is a point in the simplex with vertices C^{d-l}, $l \in I^-$. By (4.4.11), $S^+/S^- > 1$, so that the sum on the left-hand side must be in the interior of $\mathcal{L}^p$. But this sum is a point in the simplex with vertices C^{d-l}, $l \in I^+$. The claim follows.

As in our theorem, we will assume from now on that $d \geq 2$. With respect to a given orthonormal basis $\{X_s\}_{s=1}^n \subset \mathcal{G}$, we have

$$f = \frac{1}{\lambda_p} \triangle^{S^m} f = -\frac{1}{\lambda_p} \sum_{s=1}^{n} X_s^2 f.$$

Taking components, we obtain $V_f \subset V_{\hat{f}^2}$. Thus, by Theorem 2.3.5 and (4.4.9), the simplex with vertices $C^0, \ldots, C^d$ is contained in the boundary of $\mathcal{L}^p$. This, however, contradicts the previous claim. We conclude that C^d

must be in the interior of $\mathcal{L}^p$. Thus $d_f \leq d$ and therefore Theorem 4.4.4 follows.

4.5 Lower Bounds for the Range Dimension, Part II

As noted in the previous section, for $G = SU(2)$, the Casimir eigenvalue on the $SU(2)$-module W_p is $p(p+2)$ (see also Problem 4.4). By (4.2.1), the eigenvalue of $\mathcal{A}_{3,p}$ on R'_{4l}, $l = 1, \ldots, [p/2]$, within $(\mathcal{E}_3^p)^{SU(2)} = \sum_{l=1}^{[p/2]} R'_{4l}$ is $1 - \frac{4l(2l+1)}{p(p+2)}$. If, for a full $SU(2)$-equivariant p-eigenmap $f : S^3 \to S_V$, $\langle f \rangle$ is in one of the irreducible components of $(\mathcal{E}_3^p)^{SU(2)}$, say R'_{4l}, $l = 1, \ldots, [p/2]$, then, by Theorem 4.4.1:

$$\dim V \geq \frac{\dim \mathcal{H}^p}{\dim SU(2)} = \frac{(p+1)^2}{3}, \text{ if } p(p+2) \geq 4l(2l+1), \tag{4.5.1}$$

and

$$\dim V \geq \frac{\dim \mathcal{H}^p}{\dim SU(2)+1} = \frac{(p+1)^2}{4} \text{ if } p(p+2) < 4l(2l+1). \tag{4.5.2}$$

In addition, being an $SU(2)$-submodule of $(p+1)R_p$, $\dim V$ is divisible by $p+1$.

Example 4.5.1. The eigenvalue of $\mathcal{A}_{3,p}$ on $R'_{[p/2]} \subset (\mathcal{E}_3^p)^{SU(2)}$ is $\frac{2}{p+2} - 1$ if p is even, and $\frac{6}{p+2} - 1$ if p is odd. The eigenvalue is positive for $p = 3$, and negative in all the other cases. We obtain the following: (a) For a (full) cubic $SU(2)$-equivariant eigenmap $f : S^3 \to S_V$, we have $\dim V \geq 8$. (b) For a (full) $SU(2)$-equivariant minimal immersion $f : S^3 \to S_V$ of degree p and order of isotropy $[p/2]-1$, (4.5.2) holds with $p+1 \mid \dim V$. In particular, for quadratic eigenmaps the lower estimate $\dim V \geq 3$ is sharp, and the equality is realized by the Hopf map. For quartic minimal immersions, $\dim V \geq 10$ and the equality is attained by the minimal immersion $\mathcal{I}$ introduced in Section 3.6. Similarly, for quintics, $\dim V \geq 12$ holds.

Example 4.5.2. The first interesting case for multiple eigenvalues arises for degree 6 $SU(2)$-equivariant minimal immersions. (For quartic eigenmaps, see Problem 4.6.)

The operator $\mathcal{A}_{3,6}$ acting on $(\mathcal{F}_3^6)^{SU(2)} = R'_8 \oplus R'_{12}$ has two eigenvalues; the eigenvalue on R'_8 is positive $(1/6)$, and on R'_{12} it is negative $(-3/4)$. Let $f : S^3 \to S_V$ be a full $SU(2)$-equivariant minimal immersion of degree 6. If $\langle f \rangle \in R'_8$ then, by (4.5.1), $\dim V \geq 21$, and if $\langle f \rangle \in R'_{12}$ then, by (4.5.2), $\dim V \geq 14$, where we also took into account divisibility of $\dim V$ by 7. In any case, if $\langle f \rangle$ is in the convex hull of the linear slices of $\mathcal{M}_3^6$ by R'_8 and R'_{12} then $\dim V \geq 14$. On the other hand, the tetrahedral minimal immersion

Tet : $S^3 \to S^6$ introduced in Section 1.5 has (linear) range dimension 7. We obtain that $\langle Tet \rangle$ is *not* in the convex hull of the linear slices of $\mathcal{M}_3^6$ by its $SU(2)$-irreducible components. We discuss this "bulging out phenomenon" below. Note the sharp contrast with Corollary 3.6.4.

Let p be even, and $f : S^3 \to S_V$ a full $SU(2)$-equivariant minimal immersion of degree p. Let $\mathcal{A}_f \subset (\mathcal{F}_3^p)^{SU(2)}$ denote the smallest $\mathcal{A}_{3,p}$-invariant linear subspace containing $\langle f \rangle$. Since the eigenvalues of $\mathcal{A}_{3,p}$ on the $(p/2-1)$ irreducible $SU(2)'$-components of $(\mathcal{F}_3^p)^{SU(2)}$ are distinct, we have

$$\dim \mathcal{A}_f \leq \left(\frac{p}{2} - 1\right).$$

The intersection of $\mathcal{A}_f$ with $(\mathcal{M}_3^p)^{SU(2)}$ admits a simpler geometric description than the whole moduli. In our next result we will describe these $\mathcal{A}_{3,p}$-invariant slices of $(\mathcal{M}_3^p)^{SU(2)}$ for the tetrahedral *Tet* : $S^3 \to S_{R_6}$ and the octahedral *Oct* : $S^3 \to S_{R_8}$ minimal immersions introduced in Section 1.5.

Theorem 4.5.3. *(a) The linear slice of* $(\mathcal{M}_3^6)^{SU(2)}$ *with* $\mathcal{A}_{Tet}$ *is 2-dimensional, and it is an isosceles triangle with vertex at* $\langle Tet \rangle$ *and an endpoint of the base at* $\langle \widehat{Tet} \rangle$ *(as shown in Figure 30). The triangle has the property that* base/side $= \sqrt{2}$. *The other endpoint corresponds to a minimal immersion with range* $3R_6$. *The interior points of the sides correspond to minimal immersions with range* $4R_6$, *while the points in the interior of the base to range* $6R_6$.

(b) The linear slice of $(\mathcal{M}_3^8)^{SU(2)}$ *by* $\mathcal{A}_{Oct}$ *is three-dimensional and it is a tetrahedron with one vertex at* $\langle Oct \rangle$ *and another at* $\langle \widehat{Oct} \rangle$ *with range* $3R_8$ *(as shown in Figure 31). The other two vertices correspond to ranges* $2R_8$ *and* $3R_8$. *As in (a), points in the interior of the edges correspond to ranges with trivial intersection of the space of components in the sense of Theorem 2.3.5. (See also the additive property of the range dimensions in the tables at the end of this section.) Finally,* R_8', R_{12}' *and* R_{16}' *intersect three faces of the tetrahedron in points that correspond to ranges* $7R_8$, $6R_8$ *and* $6R_8$.

Inspection of how the triangle and the tetrahedron are situated with respect to the $SU(2)$-irreducible components of $\mathcal{F}_3^6$ and $\mathcal{F}_3^8$ gives the following:

Corollary 4.5.4. *There exists a full* $SU(2)$*-equivariant isotropic minimal immersion* $S^3 \to S^{27}$ *of degree* 6 *and order of isotropy* 2. *Also there exist full* $SU(2)$*-equivariant isotropic minimal immersions* $S^3 \to S^{53}$ *of degree* 8 *and order of isotropy* 2 *and* 3.

Remark 1. Based on analogy, in Theorem 4.5.3 it may seem reasonable to expect that $\mathcal{A}_{Iso}$ is a four-dimensional space whose intersection with $(\mathcal{M}_3^{12})^{SU(2)}$ is a pentatope. Recently, Weingart [1] showed, however, that $\mathcal{A}_{Iso} \cap \mathcal{M}_3^{12}$ is a (three-dimensional) tetrahedron.

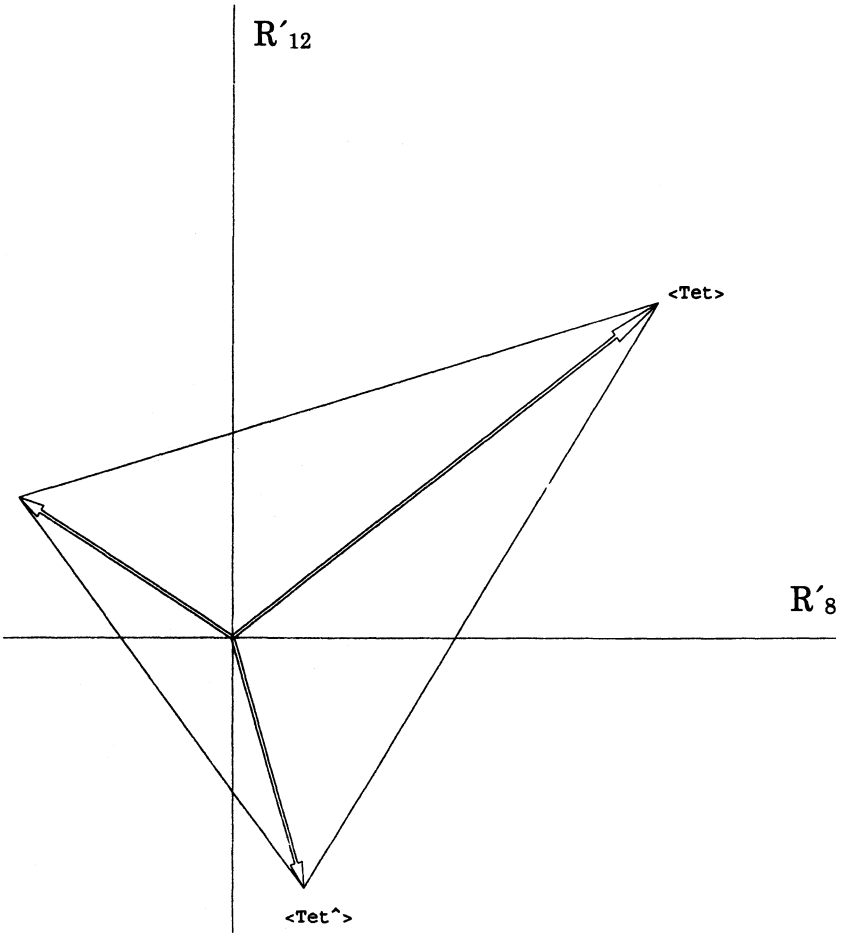


Figure 30.

Remark 2. Another recent result of Escher-Weingart [1] asserts that any $SU(2)$-equivariant minimal immersion $f : S^3 \to S_V$ of degree p with binary icosahedral invariance group (such as the icosahedral minimal immersion $Ico : S^3 \to S^{12}$) is *isotropic of order 2.* Their argument is as follows: We consider the scalar product

$$\langle \beta(f)(X_1, X_2), \beta(f)(X_3, X_4) \rangle,$$

where $\beta(f)$ is the second fundamental form of f, and $X_1, X_2, X_3, X_4 \in T(S^3)$. Since f is $SU(2)$-equivariant, we can restrict this scalar product to $T_1(S^3) = su(2)$. Varying the vectorial arguments, this scalar product defines an element of $S^2(S_0^2(su(2)))$, where the subscript is because $\beta(f)$ is traceless. We now make use of the additional fact that the icosahedral group, acting on $su(2)$ by the adjoint action, fixes this element. (This is

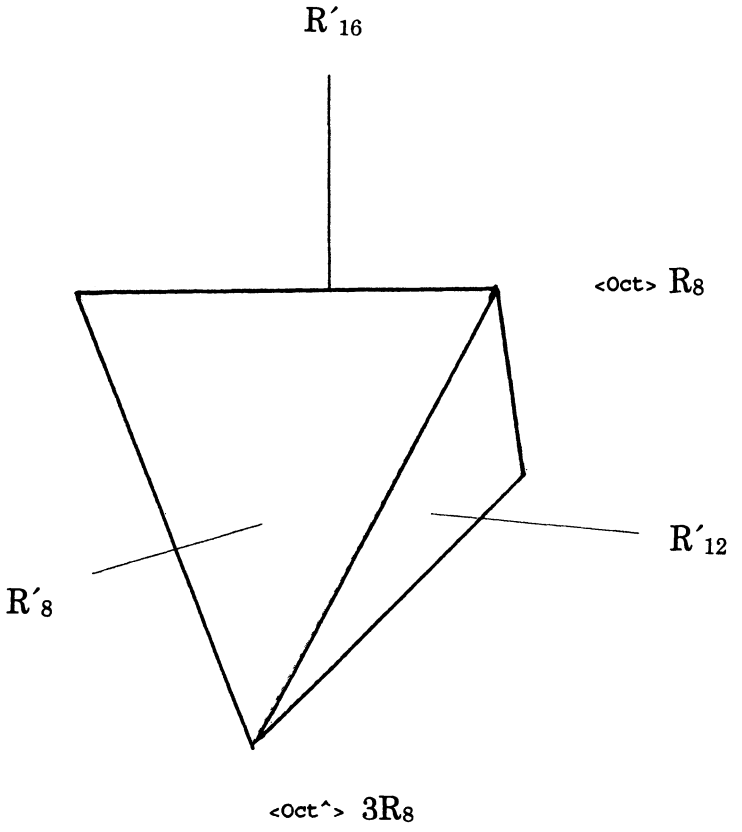


Figure 31.

because f, and hence its second fundamental form, are well-defined on the icosahedral manifold S^3/I^*.) We obtain

$$S^2(S_0^2(su(2))) \otimes \mathbf{C} \cong S^2(S_0^2(R_2)) \otimes \mathbf{C}$$
$$= S^2(R_4) \otimes \mathbf{C} = W_8 \oplus W_4 \oplus W_0.$$

Our element must lie in the trivial component W_0 since I^* has no nontrivial fixed points in W_8 and W_4. (According to Section 1.3, the smallest absolute icosahedral invariant is of degree 12.) The same argument applies when f is replaced by the standard minimal immersion f_p. Thus, the elements in $S^2(S_0^2(su(2)))$ corresponding to f and f_p are constant multiples of each other. Finally, the constant must be one, since the Gauss equations must be satisfied. We obtain that the scalar product above is the same for f and f_p. Isotropy of order 2 follows.

We recall again the equivariant construction introduced in Sections 1.4-1.5. We will use the notations introduced there. In particular, we have the standard basis $X, Y, Z \in su(2)$ given in (1.4.12). Recall that for $U \in su(2)$, we denote by U_L and U_R the left- and right-invariant extensions of U to

S^3. We have, as differential operators:

$$\begin{aligned} X_L &= -\bar{w}\partial_z - w\partial_{\bar{z}} + \bar{z}\partial_w + z\partial_{\bar{w}}, \\ Y_L &= i(-\bar{w}\partial_z + w\partial_{\bar{z}} + \bar{z}\partial_w - z\partial_{\bar{w}}), \\ Z_L &= i(z\partial_z - \bar{z}\partial_{\bar{z}} + w\partial_w - \bar{w}\partial_{\bar{w}}), \end{aligned}$$

and

$$\begin{aligned} X_R &= w\partial_z + \bar{w}\partial_{\bar{z}} - z\partial_w - \bar{z}\partial_{\bar{w}}, \\ Y_R &= i(w\partial_z - \bar{w}\partial_{\bar{z}} + z\partial_w - \bar{z}\partial_{\bar{w}}), \\ Z_R &= i(z\partial_z - \bar{z}\partial_{\bar{z}} - w\partial_w + \bar{w}\partial_{\bar{w}}). \end{aligned}$$

Lemma 4.5.5. *For any $U \in su(2)$, we have*

$$U_L \cdot f_\xi = -f_{U_R \cdot \xi}.$$

PROOF. This is a consequence of the identity

$$f_\xi \circ R_g = f_{\xi \circ L_{g^{-1}}}, \quad g \in SU(2).$$

Using standard notation, for $h \in SU(2)$, we compute

$$\begin{aligned} (U_L f_\xi)(h) &= \frac{d}{dt}|_{t=0} f_\xi(R_{\exp(tU)}(h)) \\ &= \frac{d}{dt}|_{t=0} f_{\xi \circ L_{\exp(-tU)}}(h) \\ &= -f_{U_R \xi}(h). \end{aligned}$$

Remark 1. Since $\gamma \circ L_g \circ \gamma = R_{\gamma(g)}$, $g \in SU(2)$, the left- and right-invariant vector fields can be obtained from one another by applying γ. In fact, we have

$$\gamma_*(X_R) = X_L, \quad \gamma_*(Y_R) = -Y_L, \quad \gamma_*(Z_R) = Z_L.$$

Remark 2. At $1 \in S^3$, we have $f_\xi(1) = \xi$ so that, by Lemma 4.5.5, $(U_L)_1 f_\xi = -U_R \cdot \xi$, $U \in su(2)$. We used this in Section 1.4 to derive (1.4.13) for the condition of homothety of f_ξ.

Remark 3. Escher and Weingart [1] gave a necessary and sufficient condition for an $SU(2)$-equivariant minimal immersion $f_\xi : S^3 \to S_{R_p}$ (obtained from the equivariant construction) to be isotropic (of any order) in terms of ξ.

Lemma 4.5.5 provides a particularly simple way to express $\widehat{f_\xi}$ in terms of ξ:

Corollary 4.5.6. *With respect to the standard basis $X, Y, Z \in su(2)$, we have*

$$\widehat{f_\xi} = -\frac{1}{\sqrt{\lambda_p}}\left(f_{X_R \xi}, f_{Y_R \xi}, f_{Z_R \xi}\right), \tag{4.5.3}$$

where, as operators acting on W_p:

$$X_R = w\partial_z - z\partial_w$$
$$Y_R = i(w\partial_z + z\partial_w)$$
$$Z_R = i(z\partial_z - w\partial_w).$$

From now on, we write $\widehat{f_\xi} = f_{\widehat{\xi}}$, where

$$\widehat{\xi} = -\frac{1}{\sqrt{\lambda_p}}(X_R\xi, Y_R\xi, Z_R\xi).$$

We will also use this notation if ξ has several components, i.e. if ξ is an element in some multiples of W_p or R_{2d}.

Let $p = 2d$ be even. We will use the orthonormal basis $\{\xi_j\}_{j=1}^{2d+1}$ in $R_{2d} \subset W_{2d}$ given in (1.4.6).

Lemma 4.5.7. *The 2d-eigenmap $f_{2d} : S^3 \to S_{(2d+1)R_{2d}}$ defined by*

$$f_{2d} = \frac{1}{\sqrt{2d+1}}(f_{\xi_1}, f_{\xi_2} \ldots, f_{\xi_{2d+1}}), \tag{4.5.4}$$

is congruent to the standard minimal immersion $f_{3,2d}$.

PROOF. The statement is a consequence of Schur's orthogonality relations, since the components of f_{2d}, with respect to the orthonormal basis (1.4.6) in R_{2d}, are precisely the matrix elements of the $SU(2)$-module R_{2d} (cf. also Problem 2.3). Here we give another proof based on the fact that a full $SO(m+1)$-equivariant p-eigenmap $f : S^m \to S_V$ is standard. ($\mathcal{E}^p$ has no trivial component, cf. Corollary 2.3.4.) f_{2d} defined in (4.5.4) is clearly $SU(2)$-equivariant since its components are. Since $SO(4) = SU(2) \cdot SU(2)'$, it remains to be shown that f_{2d} is also $SU(2)'$-equivariant. Recall that $SU(2)'$ is obtained from $SU(2)$ by conjugation with the diagonal matrix γ with diagonal elements $1, 1, 1, -1$. The stated $SU(2)'$-equivariance follows from the (easily verifiable) formula

$$f_\xi \circ (\gamma \circ L_{g^{-1}} \circ \gamma) = f_{\xi \circ L_{\gamma(g)}}, \quad g \in SU(2).$$

(Indeed, by a remark above, $\gamma \circ L_g \circ \gamma = R_{\gamma(g)}$, $g \in SU(2)$.) We have

$$\xi_j \circ L_{\gamma(g)} = \sum_{l=1}^{2d+1} \rho_{jl}\xi_l,$$

where the matrix $(\rho_{jl})_{j,l=1}^{2d+1}$ is orthogonal. The lemma follows.

With respect to the orthonormal basis in R_{2d} above, any $\xi \in R_{2d}$ can be written as $\xi = \sum_{j=1}^{2d+1} a_j\xi_j$, $a_j \in \mathbf{R}$, $j = 1, \ldots, 2d+1$. Applying the equivariant construction to all the polynomials involved, we obtain

$$f_\xi = \sum_{j=1}^{2d+1} a_j f_{\xi_j} = \sqrt{2d+1} \sum_{j=1}^{2d+1} \frac{a_j}{\sqrt{2d+1}} f_{\xi_j}.$$

Comparing this with (4.5.4), we see that $f_\xi = Af_{2d}$, where $A : (2d+1)R_{2d} \to R_{2d}$ is a linear map. Once again, in terms of the orthonormal basis in R_{2d}, A can be written in the block form

$$A = \sqrt{2d+1}\,[a_1, a_2, \ldots, a_{2d+1}],$$

where the j-th block is a diagonal $(2d+1)\times(2d+1)$-matrix with diagonal element a_j, $j = 1, \ldots, 2d+1$. We obtain $\langle f_\xi\rangle = C = A^\top A - I \in S^2((2d+1)R_{2d})$ where the jl-th block of C is a diagonal $(2d+1)\times(2d+1)$-matrix with diagonal element $c_{jl} = (2d+1)a_j a_l - \delta_{jl}$, $j, l = 1, \ldots, 2d+1$. In what follows, we always represent our points in $\mathcal{L}_3^{2d}$ in this form. This notation also naturally extends to the case when ξ is vector-valued. In particular, it also applies to $\widehat{f}_\xi = f_{\widehat{\xi}}$. Letting $f_{\widehat{\xi}} = \widehat{A}f_{2d}$, the matrix $\widehat{A}$ and therefore $\langle\widehat{f}\rangle = \widehat{C} = \widehat{A}^\top\widehat{A} - I$ can be computed in terms of A.

We now turn to explicit examples. We first consider quadratic eigenmaps: $d = 1$. The equivariant construction applied to the polynomial $\xi_3 = izw$ gives the quadratic eigenmap $f_{\xi_3} : S^3 \to S_{R_2}$, and this must be congruent to the Hopf map (Corollary 2.7.2). Simple computation in the use of (4.5.3) gives $\widehat{f}_{\xi_3} = \frac{1}{\sqrt{2}}(f_{\xi_2}, f_{\xi_1}) : S^3 \to S_{2R_2}$ (with a component vanishing), and this is congruent to the complex Veronese map (Example 1.4.1). On the moduli, we have $\widehat{C} = -\frac{1}{2}C$. This is in agreement with the fact that $\mathcal{A}_{3,2}$ has eigenvalue $-\frac{1}{2}$ on $(\mathcal{E}_3^2)^{SU(2)} = R_4'$.

We skip the quartic minimal immersions as they have been treated thoroughly in Section 3.6. We now put $d = 3$ and study the action of $\mathcal{A}_{3,6}$ on $(\mathcal{F}_3^6)^{SU(2)} = R_8' \oplus R_{12}'$. The eigenvalues of $\mathcal{A}_{3,6}$ on R_8' and on R_{12}' are $1/6$ and $-3/4$. In terms of the chosen orthonormal basis in R_6, the tetrahedral minimal immersion *Tet* corresponds, by the equivariant construction, to $\xi_{Tet} = \xi_3$. Using (4.5.3), we obtain

$$\widehat{\xi}_{Tet} = \left(\frac{1}{2\sqrt{2}}\xi_1 - \frac{\sqrt{30}}{12}\xi_5, \frac{1}{2\sqrt{2}}\xi_2 + \frac{\sqrt{30}}{12}\xi_6, \frac{1}{\sqrt{3}}\xi_4\right).$$

The corresponding matrices are

$$A = \sqrt{7}[0, 0, 1, 0, 0, 0, 0]$$

and

$$\widehat{A} = \sqrt{7}\begin{bmatrix} \frac{1}{2\sqrt{2}} & 0 & 0 & 0 & -\frac{\sqrt{30}}{12} & 0 & 0 \\ 0 & \frac{1}{2\sqrt{2}} & 0 & 0 & 0 & \frac{\sqrt{30}}{12} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \end{bmatrix}.$$

C and $\widehat{C}$ are linearly independent and they span the $\mathcal{A}_{3,6}$-invariant plane $\mathcal{A}_{Tet}$. Using C and $\widehat{C}$ as a basis, the intersection of $\mathcal{A}_{Tet}$ with $\mathcal{M}_3^6$ consists of those linear combinations $uC + v\widehat{C}$ for which $uC + v\widehat{C} + I$ is positive

semidefinite. This relation is easily resolved[1] in terms of u and v, and we obtain that the intersection is the isosceles triangle with vertices C, $\widehat{C}$ and $C' = -\frac{1}{3}C - \widehat{C}$. (The three sides of the triangle are determined by $\det(uC + v\widehat{C} + I) = 0$.) The matrix $\widehat{\widehat{C}}$ is in the interior of the triangle. The assertion about the range of the corresponding minimal immersions follows by working out the rank of $uC + v\widehat{C} + I$ for the vertices and (open) edges of the triangle. (a) of Theorem 4.5.3 follows. As for Corollary 4.5.4, we see that the point $\frac{1}{9}C - \frac{2}{3}\widehat{C}$ is on the side of the triangle with vertices C and C', and it is an eigenvector of $\mathcal{A}_{3,6}$ with eigenvalue $-3/4$, it therefore belongs to R'_{12}. The corresponding minimal immersion $f : S^3 \to S_{4R_6}$ is therefore isotropic of order 2. The explicit form of f can be obtained by working out $\sqrt{\frac{1}{9}C - \frac{2}{3}\widehat{C} + I}$ and precomposing it with the standard minimal immersion f_6 (cf. the proof of Theorem 2.3.1).

We finally now turn to (b) of Theorem 4.5.3. Letting $d = 4$, we have

$$(\mathcal{F}_3^8)^{SU(2)} = R'_8 \oplus R'_{12} \oplus R'_{16}$$

and the eigenvalues of $\mathcal{A}_{3,8}$ on the terms of the right-hand side are $1/2$, $-1/20$, and $-4/5$. In terms of the chosen orthonormal basis in R_8, the octahedral minimal immersion *Oct* corresponds to

$$\xi_{Oct} = \frac{\sqrt{5}}{2\sqrt{3}}\xi_1 + \frac{\sqrt{7}}{2\sqrt{3}}\xi_9.$$

Using (4.5.3), we have

$$\widehat{\xi}_{Oct} = \left(-\frac{\sqrt{2}}{4\sqrt{3}}\xi_3 + \frac{\sqrt{14}}{4\sqrt{3}}\xi_7, -\frac{\sqrt{2}}{4\sqrt{3}}\xi_4 - \frac{\sqrt{14}}{4\sqrt{3}}\xi_8, -\frac{1}{\sqrt{3}}\xi_2\right).$$

Applying (4.5.3) to this once more, we obtain

$$\begin{aligned}\widehat{\widehat{\xi}}_{Oct} = {} & \frac{1}{960\sqrt{21}} \\ & \times\Big(-336\sqrt{5}\xi_5 - 48\sqrt{35}\xi_1 + 1680\xi_9, 48\sqrt{35}\xi_2 - 336\sqrt{5}\xi_6, \\ & 72\sqrt{70}\xi_4 - 168\sqrt{10}\xi_8, 48\sqrt{35}\xi_2 + 336\sqrt{5}\xi_6, \\ & 48\sqrt{35}\xi_1 + 672\sqrt{5}\xi_5 + 1680\xi_9, 72\sqrt{70}\xi_3 + 168\sqrt{10}\xi_7, \\ & -96\sqrt{70}\xi_4, 96\sqrt{70}\xi_3, 384\sqrt{35}\xi_1\Big).\end{aligned}$$

The matrices corresponding to ξ_{Oct} and $\widehat{\xi}_{Oct}$ are

$$A = 3\left[\frac{\sqrt{5}}{2\sqrt{3}}, 0, 0, 0, 0, 0, 0, \frac{\sqrt{7}}{2\sqrt{3}}\right]$$

[1] The use of a computer algebra system is recommended.

and

$$\widehat{A} = \begin{bmatrix} 0 & 0 & -\frac{\sqrt{2}}{4\sqrt{3}} & 0 & 0 & 0 & \frac{\sqrt{14}}{4\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{2}}{4\sqrt{3}} & 0 & 0 & 0 & -\frac{\sqrt{14}}{4\sqrt{3}} & 0 \\ 0 & -\frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrices C, $\widehat{C}$ and $\widehat{\widehat{C}}$ are linearly independent, and they span the $\mathcal{A}_{3,8}$-invariant 3-space $\mathcal{A}_{Oct}$. The intersection of $\mathcal{A}_{Oct}$ with $\mathcal{M}_3^8$ is a tetrahedron[2] with vertices C, $\widehat{C}$, and

$$C' = \frac{4}{21}C - \frac{9}{7}\widehat{C} - \frac{40}{21}\widehat{\widehat{C}}$$
$$C'' = -\frac{11}{14}C + \frac{3}{7}\widehat{C} + \frac{20}{7}\widehat{\widehat{C}}.$$

In addition, $\widehat{\widehat{C}}$ is in the interior of the tetrahedron. The metric properties of the tetrahedron can be easily derived using the orthogonal decomposition $C = C_1 + C_2 + C_3$, $C_l \in R'_{4l+4}$, $l = 1, 2, 3$, and solving the system

$$C = C_1 + C_2 + C_3$$
$$\widehat{C} = \frac{1}{2}C_1 - \frac{1}{20}C_2 - \frac{4}{5}C_3$$
$$\widehat{\widehat{C}} = \frac{1}{4}C_1 + \frac{1}{400}C_2 + \frac{16}{25}C_3.$$

The range (in multiples of R_8) of the minimal immersions corresponding to the vertices, (open) edges, and (open) faces are summarized in the following tables:

Vertex	C	$\widehat{C}$	C'	C''
Range	1	3	2	3

Edge	$[C\widehat{C}]$	$[CC']$	$[CC'']$	$[\widehat{C}C']$	$[\widehat{C}C'']$	$[C'C'']$
Range	4	3	4	5	6	5

Face	$[C\widehat{C}C']$	$[C\widehat{C}C'']$	$[CC'C'']$	$[\widehat{C}C'C'']$
Range	6	7	6	8

4.6 Additional Topic: Operators

We generalize here the degree-raising and -lowering operators as well as the operator of infinitesimal rotations. The former associate to a p-eigenmap $f : S^m \to S_V$ the $(p \pm 1)$-eigenmaps $f^{\pm} : S^m \to S_{V \otimes \mathcal{H}^1}$ (Section 2.4); and

[2]The computations are tedious but elementary; once again the use of a computer algebra system is recommended.

the latter, the p-eigenmap $\hat{f} : S^m \to S_{V\otimes\mathcal{G}^*}$, where $G \subset SO(m+1)$ is a closed subgroup with Lie algebra $\mathcal{G}$ transitive on S^m (Section 4.1). The basic observation here is that (the components of) $f^{\pm}$ and $\hat{f}$ are essentially given by $SO(m+1)$-module homomorphisms $\mathcal{D}^{\pm} : \mathcal{H}^1 \to (\mathcal{H}^p)^* \otimes \mathcal{H}^{p\pm1}$, and a G-module homomorphism $\hat{\mathcal{D}} : \mathcal{G} \to (\mathcal{H}^p)^* \otimes \mathcal{H}^p$ (see Examples 4.6.3-4.6.4 below). (In the tensor product $(\mathcal{H}^p)^* \otimes \mathcal{H}^q$, it is convenient to keep $\mathcal{H}^p$ and its dual $(\mathcal{H}^p)^*$ separate.)

To generalize this observation, we let W be an orthogonal G-module, where $G \subset SO(m+1)$ is a closed subgroup acting transitively on S^m, and consider a homomorphism $\mathcal{D} : W \to (\mathcal{H}^p)^* \otimes \mathcal{H}^q$ of G-modules. We call $\mathcal{D}$ an *operator*. For $a \in W$, $\mathcal{D}_a : \mathcal{H}^p \to \mathcal{H}^q$ is a linear map, and, being a homomorphism of G-modules, $\mathcal{D}$ satisfies

$$\mathcal{D}_{g\cdot a} = g \cdot \mathcal{D}_a = \rho_p(g) \circ \mathcal{D}_a \circ \rho_p(g)^{-1}, \quad g \in G,$$

where $\rho_p : SO(m+1) \to SO(\mathcal{H}^p)$ is the $SO(m+1)$-module structure on $\mathcal{H}^p$.

The transpose of $\mathcal{D}$ viewed as a linear map $\mathcal{H}^p \otimes W \to \mathcal{H}^q$ is a G-module homomorphism $\iota = \iota^{\mathcal{D}} : \mathcal{H}^q \to \mathcal{H}^p \otimes W$ given by

$$\iota^{\mathcal{D}}(\chi') = \sum_{s=1}^{n} \mathcal{D}_{w_s}^{\top} \chi' \otimes w_s, \quad \chi' \in \mathcal{H}^q,$$

where $\{w_s\}_{s=1}^n \subset W$ an orthonormal basis.

We also have

$$(\iota^{\mathcal{D}})^{\top}(\chi \otimes a) = \mathcal{D}_a \chi, \quad \chi \in \mathcal{H}^p.$$

An operator $\mathcal{D}$ is called *metric* if $\iota^{\mathcal{D}}$ is an isometric imbedding, or equivalently, if $(\iota^{\mathcal{D}})^{\top} \circ \iota^{\mathcal{D}}$ is the identity on $\mathcal{H}^q$. In any case, $(\iota^{\mathcal{D}})^{\top} \circ \iota^{\mathcal{D}}$ is constant on each irreducible component of $\mathcal{H}^q|_G$. In particular, if $G = SO(m+1)$ then, up to a constant multiple, any operator is metric; this is because $\mathcal{H}^q$ is irreducible as an $SO(m+1)$-module.

Let $\mathcal{D} : W \to (\mathcal{H}^p)^* \otimes \mathcal{H}^q$ be a metric operator. Given a p-eigenmap $f : S^m \to S_V$, we define $f^{\mathcal{D}} : S^m \to V \otimes W^*$ by

$$f^{\mathcal{D}}(a) = \mathcal{D}_a f, \quad a \in W.$$

Here, for $a \in W$, $\mathcal{D}_a$ acts on the vector-valued spherical harmonic f in a natural way: $\alpha \circ \mathcal{D}_a f = \mathcal{D}_a(\alpha \circ f)$, $\alpha \in V^*$. In terms of an orthonormal basis $\{w_s\}_{s=1}^n \subset W$ and its dual basis $\{w_s^*\}_{w=1}^n \subset W^*$, we have

$$f^{\mathcal{D}} = \sum_{s=1}^{n} \mathcal{D}_{w_s} f \otimes w_s^*.$$

Our present problem is to study under what conditions will $f^{\mathcal{D}}$ map into the unit sphere $S_{V\otimes W^*}$, and when will $f^{\mathcal{D}}$ be a spherical minimal immersion assuming that f is. Our first lemma asserts that, for metric $\mathcal{D}$, $(f_p)^{\mathcal{D}} : S^m \to \mathcal{H}^p \otimes W^*$ (made full) is congruent to $f_q : S^m \to S_{\mathcal{H}^q}$.

Lemma 4.6.1. *Let $\mathcal{D} : W \to (\mathcal{H}^p)^* \otimes \mathcal{H}^q$ be a metric operator. Then we have*

$$(f_p)^{\mathcal{D}}(x) = \iota^{\mathcal{D}}(f_q(x)), \quad x \in S^m. \tag{4.6.1}$$

PROOF. Let $\{f_p^j\}_{j=0}^{N(p)} \subset \mathcal{H}^p$ be an orthonormal basis. With the previous notations, we compute

$$\begin{aligned}
(f_p)^{\mathcal{D}}(x) &= \sum_{s=1}^{n} \sum_{j=0}^{N(p)} (\mathcal{D}_{w_s} f_p^j)(x) \cdot f_p^j \otimes w_s^* \\
&= \sum_{s=1}^{n} \sum_{j,l=0}^{N(p)} f_q^l(x) \langle \mathcal{D}_{w_s} f_p^j, f_q^l \rangle f_p^j \otimes w_s^* \\
&= \sum_{s=1}^{n} \sum_{l=0}^{N(p)} f_q^l(x) \mathcal{D}_{w_s}^{\top}(f_q^l) \otimes w_s^* \\
&= \sum_{s=1}^{n} \mathcal{D}_{w_s}^{\top}(f_q(x)) \otimes w_s^* \\
&= \iota^{\mathcal{D}}(f_q(x)).
\end{aligned}$$

The lemma follows.

Let $f : S^m \to S_V$ be a full p-eigenmap with $\langle f \rangle = A^{\top} A - I \in S^2(\mathcal{H}^p)$, where $f = A \circ f_p$. Applying a metric operator $\mathcal{D} : W \to (\mathcal{H}^p)^* \otimes \mathcal{H}^q$ to both sides of this equation, for $x \in S^m$, we have

$$f^{\mathcal{D}}(x) = (A \otimes I)(f_p)^{\mathcal{D}}(x) = (A \otimes I)\, \iota^{\mathcal{D}}(f_q(x)),$$

where, in the last equality, we used (4.6.1). Since $\mathcal{D}$ is metric, we thus obtain

$$\langle f^{\mathcal{D}} \rangle = (\iota^{\mathcal{D}})^{\top} \circ (A^{\top} A - I) \circ \iota^{\mathcal{D}} + I = (\iota^{\mathcal{D}})^{\top} \circ (\langle f \rangle \otimes I) \circ \iota^{\mathcal{D}}.$$

This motivates us to define $\Phi^{\mathcal{D}} : S^2(\mathcal{H}^p) \to S^2(\mathcal{H}^q)$ by

$$\Phi^{\mathcal{D}}(C) = (\iota^{\mathcal{D}})^{\top} \circ (C \otimes I) \circ \iota^{\mathcal{D}}, \quad C \in S^2(\mathcal{H}^p). \tag{4.6.2}$$

By the previous computation, we have

$$\Phi^{\mathcal{D}}(\langle f \rangle) = \langle f^{\mathcal{D}} \rangle. \tag{4.6.3}$$

$\Phi^{\mathcal{D}} : S^2(\mathcal{H}^p) \to S^2(\mathcal{H}^q)$ is a homomorphism of G-modules. By (4.6.3), given a p-eigenmap $f : S^m \to S_V$, $f^{\mathcal{D}}$ will map into the unit sphere $S_{V \otimes W^*}$ and thereby $f^{\mathcal{D}}$ will become a q-eigenmap, provided that $\Phi^{\mathcal{D}}(\mathcal{E}^p) \subset \mathcal{E}^q$. In a similar vein, if f is a spherical minimal immersion, then so is $f^{\mathcal{D}}$ if $\Phi^{\mathcal{D}}(\mathcal{F}^p) \subset \mathcal{F}^q$. We arrive at the following:

Theorem 4.6.2. *Let $G \subset SO(m+1)$ be a closed subgroup acting transitively on S^m. Assume that $S^2(\mathcal{H}^r)|_G$, $r = \max(p, q)$, has multiplicity*

one decomposition into irreducible G-modules. Then, given a metric operator $\mathcal{D} : W \to (\mathcal{H}^p)^* \otimes \mathcal{H}^q$, for any minimal immersion $f : S^m \to S_V$ of degree p, $f^{\mathcal{D}} : S^m \to S_{V \otimes W^*}$ is also a minimal immersion of degree q. The statement is also true for isotropic minimal immersions, including eigenmaps.

Remark. In view of (3.1.8), for $G = SO(m+1)$, the conditions of Theorem 4.6.2 are satisfied.

Example 4.6.3. We let $G = SO(m+1)$ and $W = \mathbf{R}^{m+1}$ with its standard $SO(m+1)$-module structure given by matrix multiplication. We define $\mathcal{D}^{\pm} : \mathbf{R}^{m+1} \to (\mathcal{H}^p)^* \otimes \mathcal{H}^{p\pm 1}$ by

$$\mathcal{D}^+_a = \sqrt{\frac{2p+m-1}{p+m-1}}\,\delta_a,$$

and

$$\mathcal{D}^-_a = \frac{1}{\sqrt{p(2p+m-1)}}\,\partial_a, \quad a \in \mathbf{R}^{m+1},$$

where $\partial_a : \mathcal{H}^p_m \to \mathcal{H}^{p-1}_m$ is partial differentiation in the direction $a \in \mathbf{R}^{m+1}$, and $\delta_a = H(a^* \cdot)$ (cf. Section 2.1). $\mathcal{D}^{\pm}$ are the degree raising and lowering operators introduced in Section 2.4. In particular, $\iota_{\pm} = \iota^{\mathcal{D}^{\pm}}$ and $\Phi^{\pm}_p = \Phi^{\mathcal{D}^{\pm}}$. By (2.1.18), $\mathcal{D}^{\pm}$ are both metric.

Example 4.6.4. In degree-raising and -lowering, we set $G = SO(m+1)$. In our next example, we let $G \subset SO(m+1)$ be a closed subgroup with Lie algebra $\mathcal{G}$, and we assume that G acts on S^m transitively. We let $W = \mathcal{G}$ be the G-module with the adjoint representation, and $\hat{\mathcal{D}} : \mathcal{G} \to (\mathcal{H}^p)^* \otimes \mathcal{H}^p$ the induced action of $\mathcal{G}$ on $\mathcal{H}^p$. Inserting the factor of $1/\lambda_p$ in the definition of $\hat{\mathcal{D}}$, it becomes the operator of infinitesimal rotations introduced in Section 4.1. We have $\alpha_p = \iota^{\hat{\mathcal{D}}}$. By Lemma 4.1.2, $\hat{\mathcal{D}}$ is metric. $\Phi^{\hat{\mathcal{D}}}$ in (4.6.2) specializes to $\mathcal{A}_p$, where $\mathcal{A}_p(C) = \alpha_p^{\top} \circ (C \otimes I) \circ \alpha_p$, $C \in S^2(\mathcal{H}^p)$. As noted in Section 4.4, $\mathcal{A}_p$ is a symmetric endomorphism of $S^2(\mathcal{H}^p)$ that may well vanish on some irreducible components. By Theorem 4.2.1, we have $\mathcal{A}_p(\mathcal{E}^p_m) \subset \mathcal{E}^p$, and $\mathcal{A}_p(\mathcal{F}^{p,k}) \subset \mathcal{F}^{p;k}$, in particular, $\mathcal{A}_p$ restricts to self-maps on the moduli $\mathcal{L}^p$ and $\mathcal{M}^p$.

For the rest of this section we assume that $G = SO(m+1)$. For $(u,v) \in \triangle^{p,q}_0$, $u+v \equiv p+q \,(\mathrm{mod}\, 2)$, we consider the irreducible component $V^{(u,v,0,\ldots,0)} \subset \mathcal{H}^p \otimes \mathcal{H}^q$ in the decomposition (3.1.7). Up to scaling, this inclusion defines a metric operator $D^{u,v} : V^{(u,v,0,\ldots,0)} \to (\mathcal{H}^p)^* \otimes \mathcal{H}^q$ since $(\mathcal{H}^p)^* \cong \mathcal{H}^p$. In what follows, we describe $D^{u,v}$ in terms of polynomials in the two sets of commuting variables ∂_i and δ_i, $i = 0, \ldots, m$. (Since the representations that occur here are absolutely irreducible, we work over $\mathbf{C}$ and adjust the notation accordingly.)

Let

$$\mathcal{P}^{q,p} = \mathcal{P}^q[y_0, \ldots, y_m] \otimes \mathcal{P}^p[x_0, \ldots, x_m]$$

and define

$$\Xi : \mathcal{P}^{q,p} \to (\mathcal{H}^p)^* \otimes \mathcal{H}^q$$

as follows: If $\xi \in \mathcal{P}^{q,p}$ then $\Xi(\xi) : \mathcal{H}^p \to \mathcal{H}^q$ is given by

$$\Xi(\xi) = \xi(\delta_0, \ldots, \delta_m; \partial_0, \ldots, \partial_m),$$

where $\xi = \xi(y_0, \ldots, y_m; x_0, \ldots, x_m)$. (Note that $[\partial_i, \partial_k] = [\delta_i, \delta_k] = 0$, so that this definition makes sense.) Ξ is a homomorphism of $SO(m+1)$-modules.

Proposition 4.6.5. *Ξ is surjective.*

PROOF. Let $L : \mathcal{H}^p \to \mathcal{H}^q$ be linear and assume that L is orthogonal to the image of Ξ. Fix $0 \le i_1, \ldots, i_p; k_1, \ldots, k_q \le m$. For $\xi = y_{k_1} \ldots y_{k_q} \otimes x_{i_1} \ldots x_{i_p}$, we have

$$\langle \Xi(\xi), L \rangle = \sum_{j=0}^{N(p)} \sum_{l=0}^{N(q)} \langle \Xi(\xi) f_p^j, f_q^l \rangle \langle L f_p^j, f_q^l \rangle = 0.$$

By Lemma 2.1.4, the transpose of δ_i is, up to a constant multiple, ∂_i. Thus, we obtain

$$\sum_{j=0}^{N(p)} \sum_{l=0}^{N(q)} \langle L f_p^j, f_q^l \rangle \frac{\partial^p f_p^j}{\partial x_{i_1} \ldots \partial x_{i_p}} \frac{\partial^q f_q^l}{\partial y_{k_1} \ldots \partial y_{k_q}} = 0.$$

Using Lemma 2.1.4 again, $\partial^p f_p^j / \partial x_{i_1} \ldots \partial x_{i_p}$ is a constant multiple of $\langle f_p^j, H(x_{i_1} \ldots x_{i_p}) \rangle$. We arrive at

$$\langle L(H(x_{i_1} \ldots x_{i_p})), H(x_{k_1} \ldots x_{k_q}) \rangle = 0.$$

Finally, the polynomials $H(x_{i_1} \ldots x_{i_p})$ span $\mathcal{H}^p$ and $L = 0$ follows.

Consider the canonical decomposition

$$\mathcal{P}^q[y_0, \ldots, y_m] = \mathcal{H}^q \oplus \rho^2 \mathcal{P}^{q-2}[y_0, \ldots, y_m], \qquad \rho^2 = y_0^2 + \ldots y_m^2.$$

For $\xi = \rho^2 \xi' \in \mathcal{P}^{q,p}$, $\xi' \in \mathcal{P}^{q-2,p}$, we have $\Xi(\xi) = \Xi(\rho^2 \xi') = H(\rho^2 \Xi(\xi')) = 0$, so that $\rho^2 \mathcal{P}^{q-2}[y_0, \ldots, y_m] \otimes \mathcal{P}^p[x_0, \ldots, x_m]$ is in the kernel of Ξ. In the x variables, ρ^2 corresponds to the Laplacian so that $\mathcal{P}^q[y_0, \ldots, y_m] \otimes \rho^2 \mathcal{P}^{p-2}[x_0, \ldots, x_m]$ is also in the kernel of Ξ. Factoring out, we obtain that

$$\Xi : \mathcal{H}^q \otimes \mathcal{H}^p \to (\mathcal{H}^p)^* \otimes \mathcal{H}^q$$

is an isomorphism of $SO(m+1)$-modules.

We now define

$$\mathcal{D}^{u,v} = \Xi|_{V^{(u,v,0,\ldots,0)}} : V^{(u,v,0,\ldots,0)} \to (\mathcal{H}^p)^* \otimes \mathcal{H}^q,$$

where $V^{(u,v,0,\dots,0)}$ is considered as a component of $\mathcal{H}^q \otimes \mathcal{H}^p$. It follows that, for each $\xi \in V^{(u,v,0,\dots,0)}$, $\mathcal{D}^{u,v}_\xi$ is a polynomial in the operators δ_i and ∂_i, $i = 0, \dots, m$, homogeneous of degree q in the δ_i's and degree p in ∂_i's. Clearly, $\mathcal{D}^{u,v}$ defines an operator.

Remark. The operators $D^{u,v}$ can be described in terms of the Young tableau of $V^{(u,v,0,\dots,0)} \subset \mathcal{H}^p \otimes \mathcal{H}^q$ (cf. Appendix 3).

Problems

4.1. Work out the eigenvalues of the composition $\Phi^+ \circ \Phi^-$. Relate this to $\mathcal{A}_p$ through the decomposition

$$\mathcal{H}^1 \otimes \mathcal{H}^1 = \mathcal{H}^0 \otimes \wedge^2(\mathcal{H}^1) \otimes S^2_0(\mathcal{H}^1),$$

where $\mathcal{H}^0$ corresponds to the trace, $\wedge^2(\mathcal{H}^1) \cong so(m+1)^*$, and $S^2_0(\mathcal{H}^1) \subset S^2(\mathcal{H}^1)$ is the traceless part isomorphic with $\mathcal{H}^2$. Define the operator corresponding to the third summand. (Hint: Consider the operator $\mathcal{B}_p : S^2(\mathcal{H}^1) \to (\mathcal{H}^p)^* \otimes \mathcal{H}^p$ given by

$$\mathcal{B}_p(a^* \odot b^*) = \frac{1}{\sqrt{\lambda_p}}(\delta_a \circ \partial_b + \delta_b \circ \partial_a), \quad a, b \in \mathbf{R}^{m+1}.)$$

4.2. Let $f : S^3 \to S_{R_p}$ be an $SU(2)$-equivariant minimal immersion of degree $p \geq 6$. (f exists since the system of equations (1.4.16) is solvable for $p \geq 6$, cf. Problem 3.1.) Show that the critical exponent d_f of f satisfies

$$\binom{d_f + 3}{3} \geq p + 1.$$

4.3. Let $f : S^3 \to S_V$ be an $SU(2)$-equivariant minimal immersion of degree p and order of isotropy $k \leq [p/2] - 2$. Use Theorem 4.4.4 to show that the critical exponent of f satisfies

$$d_f \leq \left[\frac{p}{2}\right] - k.$$

4.4. Show that the Casimir eigenvalue on the $SU(2)$-module W_p is $p(p+2)$.

4.5. Derive (1.4.13) using Lemmas 4.5.5 and 4.5.7 and $SU(2)$-equivariance.

4.6. Study the bulging out phenomenon for $(\mathcal{L}^4_3)^{SU(2)}$ ($SU(2)$-equivariant quartic eigenmaps) along the lines of Example 4.5.2. (Hint: Consider $Ver_2 \circ Hopf : S^3 \to S^4$; cf. Section 1.4.)

4.7. Show that, for $m \geq 3$ and $p \geq 4$, we have

$$\frac{\dim \mathcal{H}^p}{\dim so(m+1)} < \frac{\dim \mathcal{H}^{p-1}}{\dim \mathcal{H}^1}.$$

4.8. Define the composition of operators in a natural way. Work out the q-th power of $\mathcal{D}^+$ (Example 4.6.3) using the following steps: The operator in question

should be of the form

$$(\mathcal{D}^+)^q : \mathcal{P}^q \to (\mathcal{H}^p)^* \otimes \mathcal{H}^{p+q},$$

where $\mathcal{P}^q = \mathcal{P}^q[y_0, \ldots, y_m] = \mathcal{H}^1 \otimes \ldots \otimes \mathcal{H}^1 (q \text{ times})$. Explicitly, for $\chi \in \mathcal{H}^p$:

$$(\mathcal{D}^+)^q_{y_{i_1} \ldots y_{i_q}} \chi = \sqrt{\frac{\lambda_{2p} \ldots \lambda_{2(p+q-1)}}{2^q \lambda_p \ldots \lambda_{p+q-1}}} \times H(x_{i_1} H(x_{i_2} \ldots H(x_{i_q} \chi) \ldots)). \tag{4.6.4}$$

The last term (without the constant multiple) can be written as

$$H(x_{i_1} H(x_{i_2} \ldots H(x_{i_q} \chi) \ldots)) = H(H(x_{i_1} \ldots x_{i_q}) \chi).$$

The decomposition of the tensor product $(\mathcal{H}^p)^* \otimes \mathcal{H}^{p+q}$ into irreducible components (3.1.5) shows that it has only one component, namely $\mathcal{H}^q$, common with that of $\mathcal{P}^q$. (Geometrically, the triangle $\triangle_0^{p+q,p}$ intersects the segment $[0, q]$ only at the point q.) Thus, $(\mathcal{D}^+)^q$ factors through the harmonic projection $H : \mathcal{P}^q \to \mathcal{H}^q$ giving the operator

$$(\mathcal{D}^+)^q : \mathcal{H}^q \to (\mathcal{H}^p)^* \otimes \mathcal{H}^{p+q}$$

(denoted by the same symbol) by

$$(\mathcal{D}^+)^q_{H(y_{i_1} \ldots y_{i_q})} \chi = \sqrt{\frac{\lambda_{2p} \ldots \lambda_{2(p+q-1)}}{2^q \lambda_p \ldots \lambda_{p+q-1}}} \times H(H(x_{i_1} \ldots x_{i_q}) \chi). \tag{4.6.5}$$

This can still be rewritten in terms of the standard minimal immersion $f_q : S^m \to S_{\mathcal{H}^q}$ taken in the form

$$f_q(x) = \sqrt{\frac{\lambda_2 \lambda_4 \ldots \lambda_{2(q-1)}}{2^{q-1} \lambda_1 \lambda_2 \ldots \lambda_{q-1}}} \times \sum_{i_1, \ldots, i_q = 0}^{m} H(x_{i_1} \ldots x_{i_q}) H(y_{i_1} \ldots y_{i_q}). \tag{4.6.6}$$

Combining these, arrive at the operator

$$(\mathcal{D}^+)^q : \mathcal{H}^q \to (\mathcal{H}^p)^* \otimes \mathcal{H}^{p+q}$$

given by

$$(\mathcal{D}^+)^q_{f_q(x)} \chi = \sqrt{\frac{\lambda_{2p} \ldots \lambda_{2(p+q-1)}}{(2^q \lambda_p \ldots \lambda_{p+q-1}}} H(f_q(x) \chi),$$

where $\chi \in \mathcal{H}^p$.

4.9. Define the transpose of operators in a natural way. Work out the transposes of $\mathcal{D}^{\pm}$ and $\hat{\mathcal{D}}$.

4.10. Determine the operator $\mathcal{D}^{1,0}$ acting on $V^{(1,0,\ldots,0)}$, and the operator $\mathcal{D}^{1,1}$ acting on $V^{(1,1,0,\ldots,0)}$.

4.11. Generalize the operator of infinitesimal rotations to act on eigenmaps $f : M \to S_V$, where M is a compact isotropy irreducible Riemannian homogeneous space.

4.12. **(a)** Use Problem 2.19 to show that, for a p-eigenmap $f : S^m \to S_V$, we have

$$V_{\hat{f}} \subset \delta\partial V_f.$$

(b) Let $V \subset \mathcal{H}^p$ be a linear subspace. Prove that $V \neq \mathcal{H}^p$ implies $V \neq \delta\partial V$. (Hint: If $V = \delta\partial V$ then V is an $so(m+1)$-submodule of $\mathcal{H}^p$.)

Appendix

A.1. Convex Sets

In this section we summarize some basic facts about convex sets. The principal references are Berger [1], Chapter 11, and Valentine [1].

Let $\mathcal{E}$ be a finite dimensional vector space. A set $\mathcal{L} \subset \mathcal{E}$ is convex if $C_1, C_2 \in \mathcal{L}$ implies that $tC_1 + (1-t)C_2 \in \mathcal{L}$ for all $t \in [0,1]$. Geometrically, $\mathcal{L}$ contains the line segment connecting any of its two points.

The intersection of convex sets is convex. Given a subset $\mathcal{A} \in \mathcal{E}$, the smallest convex set that contains $\mathcal{A}$ is called the *convex hull* of $\mathcal{A}$. The convex hull is denoted by $\mathrm{Hull}\,(\mathcal{A})$. Clearly, $\mathrm{Hull}\,(\mathcal{A})$ is the intersection of all convex subsets of $\mathcal{E}$ that contain $\mathcal{A}$.

Let $\mathcal{L} \subset \mathcal{E}$ be convex. A point $C \in \mathcal{L}$ is called *extremal* if C is not in the interior of a segment contained in $\mathcal{L}$. The set of extremal points of $\mathcal{L}$ is denoted by $\mathrm{Ext}\,(\mathcal{L})$. Clearly, the extremal points of $\mathcal{L}$ are on the boundary of $\mathcal{L}$.

The following result is due to Krein-Milman:

Theorem A.1.1. *A compact convex set $\mathcal{L}$ in a finite dimensional vector space $\mathcal{E}$ is the convex hull of its extremal points:*

$$\mathrm{Hull}\,(\,\mathrm{Ext}\,(\mathcal{L})) = \mathcal{L}.$$

For the proof we need some preparation. Given a subset $\mathcal{A} \subset \mathcal{E}$, and a hyperplane (a codimension-one affine subspace) $\mathcal{E}_0 \subset \mathcal{E}$, we say that $\mathcal{E}_0$ is a *supporting hyperplane* for $\mathcal{A}$ if $\mathcal{A} \cap \mathcal{E}_0$ is nonempty and one of the open half-

spaces determined by $\mathcal{E}_0$ is disjoint from $\mathcal{A}$. We say that $\mathcal{E}_0$ is a supporting hyperplane for $\mathcal{A}$ at the points of $\mathcal{A} \cap \mathcal{E}_0$. A straightforward consequence of the Hahn-Banach theorem is that any boundary point C of a closed convex set $\mathcal{L} \subset \mathcal{E}$ has a supporting hyperplane $\mathcal{E}_0$ at C. (Indeed, being a boundary point, C is disjoint from the interior of $\mathcal{L}$ and the Hahn-Banach theorem guarantees the existence of a hyperplane $\mathcal{E}_0$ that contains C but one of its open half-spaces is disjoint from the interior of $\mathcal{L}$.)

PROOF OF THEOREM A.1.1. We proceed by induction with respect to the dimension n of $\mathcal{E}$. For $n = 1$, a compact convex subset of $\mathcal{E}$ is nothing but a closed interval $\mathcal{L}$ that is clearly the convex hull of its endpoints, the extremal points of $\mathcal{L}$. For the general induction step, let $\dim \mathcal{E} = n \geq 2$ and assume that the statement is true for all compact convex subsets in vector spaces of dimension $\leq n - 1$. Let C be a boundary point of $\mathcal{L}$, and $\mathcal{E}_0$ a supporting hyperplane of $\mathcal{L}$ at C. By the definition of extremal points, we have

$$\operatorname{Ext}(\mathcal{L} \cap \mathcal{E}_0) = \operatorname{Ext}(\mathcal{L}) \cap \mathcal{E}_0. \tag{A.1.1}$$

By the induction hypothesis and (A.1.1), we have

$$\begin{aligned} C \in \mathcal{L} \cap \mathcal{E}_0 &= \operatorname{Hull}(\operatorname{Ext}(\mathcal{L} \cap \mathcal{E}_0) \\ &= \operatorname{Hull}(\operatorname{Ext}(\mathcal{L}) \cap \mathcal{E}_0) \subset \operatorname{Hull}(\operatorname{Ext}(\mathcal{L})). \end{aligned}$$

We obtain that the boundary points of $\mathcal{L}$ are in the convex hull of the extremal points of $\mathcal{L}$. On the other hand, the interior points of a convex set are obviously in the convex hull of the boundary points. We thus have $\mathcal{L} \subset \operatorname{Hull}(\operatorname{Ext}(\mathcal{L}))$. The converse is obvious since $\mathcal{L}$ is convex. Theorem A.1.1 follows.

A convex set $\mathcal{L}$ in a Euclidean vector space $\mathcal{E}$ is called a *convex body* if $\mathcal{L}$ has nonempty interior. Let $\mathcal{L}$ be a compact convex body in $\mathcal{E}$, and $O \in \mathcal{L}$ an interior point. A directed line ℓ through O meets the boundary $\partial\mathcal{L}$ at exactly two points—C and C^o (Berger [1]), where we choose the notation such that $C > C^o$. We call C^o the *antipodal* of C with respect to O. Reversing the direction on ℓ, we obtain $(C^o)^o = C$. We define the *distortion* of $\mathcal{L}$ at ℓ by

$$\lambda_O(\ell) = \frac{d(O,C)}{d(O,C^o)}, \tag{A.1.2}$$

where d is the Euclidean distance in $\mathcal{E}$. We have

$$\lambda_O(\ell^o) = \frac{1}{\lambda_O(\ell)},$$

where ℓ^o is ℓ with reverse direction. We also write $\lambda_O(\ell) = \lambda_O(C)$. With this, λ_O becomes a function on $\partial\mathcal{L}$, and $\lambda_O(C^o) = 1/\lambda_O(C)$. The distortion λ_O is continuous on $\partial\mathcal{L}$ since both the numerator and the denominator in

(A.1.2) are continuous (cf. Berger [1], p. 342). The following result is a conseequence of Helly's theorem (Berger [1], p. 366):

Theorem A.1.2. *For a compact convex body $\mathcal{L}$ in a Euclidean vector space $\mathcal{E}$, we have*

$$\frac{1}{\dim \mathcal{E}} \leq \lambda_O \leq \dim \mathcal{E}, \tag{A.1.3}$$

provided that $O \in \mathcal{L}$ is chosen appropriately.

Notice that the bounds are the best possible. For an example, consider a regular simplex $\mathcal{L}$ in $\mathcal{E}$ with O its centroid.

A function $\xi : \mathcal{L} \to \mathbf{R}$ on a convex set $\mathcal{L} \subset \mathcal{E}$ is said to be *convex* if

$$\xi(tC_1 + (1-t)C_2) \leq t\xi(C_1) + (1-t)\xi(C_2), \quad C_1, C_2 \in \mathcal{L},\ 0 \leq t \leq 1.$$

A convex function $\xi : \mathcal{L} \to \mathbf{R}$ is continuous in the interior of $\mathcal{L}$. If, in addition, ξ is continuous on the boundary of $\mathcal{L}$ then ξ attains its maximum at (at least) one extremal point.

A.2. Harmonic Maps and Minimal Immersions

The purpose of this section is to give a very brief account of some general concepts and facts on harmonic maps and minimal immersions. For proofs, see Eells-Sampson [1], Eells-Lemaire [1], or Toth [5], Chapter 1.

Let M and N be Riemannian manifolds and $f : M \to N$ a map. (All objects considered are of class C^∞.) The *differential* f_* of f at $x \in M$ is a linear map $f_{*_x} : T_x(M) \to T_{f(x)}(N)$. The *energy density* of f is the function $e(f)$ on M which, at $x \in M$, is the Hilbert-Schmidt norm square of f_{*_x}:

$$e(f) = \operatorname{trace}\{(X, Y) \mapsto \langle f_*(X), f_*(Y)\rangle\}.$$

For each vector field X on M, $f_*(X)$ is a vector field along f, i.e. a section of the pull-back vector bundle $f^*T(N)$. Thus we can view f_* as a 1-form on M with values in $f^*T(N)$. In other words, f_* is a section of the tensor product bundle $T^*(M) \otimes f^*T(N)$. This vector bundle carries a fibre metric (induced from the Riemannian metrics of M and N) and again $e(f)$ is just the norm square of f_* in this metric.

Given any precompact domain D in M we define

$$E_D(f) = \int_D e(f) \cdot v_M,$$

the *energy* of f *over* D. Here v_M is the Riemmanian volume element. For $M = D$ compact $E(f) = E_M(f)$ is the *energy of* f.

A map $f : M \to N$ is said to be *harmonic* if the energy is stable to first order with respect to (compactly supported) variations of f.

Taking a (compactly supported) vector field v along f, the *first variation formula* says

$$\delta E(v) = -2 \int_M \langle \operatorname{trace} \beta(f), v \rangle \cdot v_M,$$

where

$$\beta(f) = \nabla f_* : S^2T(M) \to f^*T(N)$$

is the *second fundamental form of* f defined by

$$\beta(f)(X,Y) = \nabla_X(f_*(Y)) - f_*(\nabla_X Y),$$

for X, Y vector fields on M. The covariant differentiation ∇ of the (Riemannian connected) vector bundle $\wedge T^*(M) \otimes f^*T(N)$ is $\nabla^M \otimes f^*\nabla^N$, where ∇^M and ∇^N are the Levi-Civita covariant differentiations of M and N. By definition, $\beta(f)$ is a symmetric 2-tensor on M with values in $f^*T(N)$, or equivalently, $\beta(f)$ is a section of $S^2T^*(M) \otimes f^*T(N)$.

$f : M \to N$ is said to be *totally geodesic* if $\beta(f)$ vanishes identically. It follows immediately from the definitions that f is totally geodesic iff it maps geodesics on M to geodesics on N, linearly.

By the first variation formula, $f : M \to N$ is a harmonic map iff

$$\operatorname{trace} \beta(f) = 0,$$

where the trace is taken pointwise on each of the tangent spaces of M.

As a specific example, let V be a Euclidean vector space with Riemannian metric imported from the scalar product on V via the natural shift $\check{}\,: T(V) \to V$. Let $f : M \to V$ be a map. For a vector field X on M, we have

$$f_*(X)\check{} = df(X) = X(f), \tag{A.2.1}$$

where X acts on f componentwise: $\alpha(X(f)) = X(\alpha \circ f)$, $\alpha \in V^*$. The Levi-Civita covariant differentiation on V gives $(\nabla_X v)\check{} = X(\check{v})$, where v is a vector field along f. The second fundamental form of $f : M \to V$ therefore reduces to

$$\beta(f)(X,Y)\check{} = X(Y(f)) - (\nabla_X Y)f.$$

Recall that the (geometric) Laplacian $\triangle^M$ acting on functions ξ on M is the trace of the bilinear form

$$(X,Y) \mapsto -X(Y(\xi)) + (\nabla_X Y)\xi.$$

Thus we have

$$\operatorname{trace} \beta(f)(X,Y)\check{} = -\triangle^M f,$$

where $\triangle^M$ acts on f componentwise: $\alpha(\triangle^M f) = \triangle^M(\alpha \circ f)$, $\alpha \in V^*$. We obtain that $f : M \to V$ is a harmonic map iff $\triangle^M f = 0$. Equivalently, $f : M \to V$ is harmonic iff the components $\alpha \circ f$, $\alpha \in V^*$, of f are harmonic functions on M.

In Section 2.1 we stated the following comparison formula between the Euclidean Laplacian $\triangle$ and the spherical Laplacian $\triangle^{rS^m}$:

$$\triangle^{rS^m}(\xi|_{rS^m}) = \left(-\triangle\xi + \frac{1}{r^2}\partial_x^2\xi + \frac{m-1}{r^2}\partial_x\xi\right)\Big|_{rS^m}$$

valid for any function ξ on $\mathbf{R}^{m+1}$, were $\partial_x = \sum_{i=0}^m x_i\partial_i$ is *radial differentiation.* (For computational simplicity, for $\mathbf{R}^{m+1}$ we use the analytic Laplacian $\triangle = \sum_{i=0}^m \partial_i^2 = -\triangle^{\mathbf{R}^{m+1}}$.) As an application of the concepts above we now derive this formula:

By definition, $(\triangle\xi)|_{rS^m}$ is the trace of the bilinear form

$$(X, Y) \mapsto X(Y(\xi)) - (\nabla_X Y)\xi.$$

Here the trace is taken on the tangent spaces of $\mathbf{R}^{m+1}$ at each point of rS^m. To evaluate this trace we choose a local orthonormal frame whose first m vector fields are tangent to rS^m, and whose last element is the radial unit vector field ν on $\mathbf{R}^{m+1} - \{0\}$ given by $\check{\nu}_x = x/|x|$, $0 \neq x \in \mathbf{R}^{m+1}$. Evaluating the bilinear form above on the first m vector fields, we obtain $-\triangle^{rS^m}\xi$ plus the trace of the correction term

$$(X, Y) \mapsto \left(\nabla_X^{rS^m} Y - \nabla_X Y\right)\xi$$

on rS^m. This correction is due to the fact that in order to evaluate the spherical Laplacian $\triangle^{rS^m}$ we need to take the Levi-Civita covariant differentiation ∇^{rS^m} on rS^m instead of ∇ on $\mathbf{R}^{m+1}$. By definition, for X, Y tangent to rS^m, $\nabla_X^{rS^m} Y$ is the projection of $\nabla_X Y$ to $T(rS^m)$. Hence the correction term reduces to

$$(X, Y) \mapsto -\langle \nabla_X Y, \nu\rangle\, \nu \cdot \xi.$$

Since X, Y are tangent to rS^m, we have

$$-\langle \nabla_X Y, \nu\rangle = \langle \nabla_X \nu, Y\rangle = \langle X(\check{\nu}), \check{Y}\rangle = \frac{1}{r}\langle \check{X}, \check{Y}\rangle = \frac{1}{r}\langle X, Y\rangle.$$

We obtain that the trace of the correction term is m/r. Summarizing and adding ν to the orthonormal frame, we arrive at the formula

$$(\triangle\xi)|_{rS^m} = -\triangle^{rS^m}(\xi|_{rS^m}) + \left(\frac{m}{r}\nu \cdot \xi + \nu(\nu \cdot \xi) - (\nabla_\nu \nu)\,\xi\right)\Big|_{rS^m}.$$

For $x \in rS^m$, we have

$$\nu_x \cdot \xi = \frac{d}{dt}\xi\left(x + t\frac{x}{|x|}\right)\Big|_{t=0} = \frac{1}{|x|}\sum_{i=0}^m x_i\partial_i\xi = \frac{1}{|x|}\partial_x\xi.$$

With this, we have

$$\nu_x(\nu \cdot \xi) = \frac{1}{|x|}\partial_x\left(\frac{1}{|x|}\partial_x\xi\right) = \frac{1}{|x|^2}\partial_x^2\xi - \frac{1}{|x|^2}\partial_x\xi.$$

Finally, we have

$$\nabla_\nu \nu = \nu(\check{\nu}) = \frac{1}{|x|}\frac{d}{dt}\left(\frac{x+tx}{|x+tx|}\right)\Big|_{t=0} = 0.$$

Restricting all the terms to rS^m amounts to setting $|x| = r$. The comparison formula follows.

Returning to the main line, assume now that N is the unit sphere S_V of a Euclidean vector space V. We can view $f : M \to S_V$ as a vector-valued function $f : M \to V$ (denoted by the same symbol) satisfying $|f|^2 = \langle f, f\rangle = 1$. In a similar vein, a vector field v along f gives rise to a map $\check{v} : M \to V$ obtained from v by applying the natural shift $\check{}: T(V) \to V$. Since v is tangent to S_V, we have $\langle \check{v}, f\rangle = 0$.

If X is a vector field on M, we have

$$(\nabla_X v)\check{} = X(\check{v}) - \langle X(\check{v}), f\rangle \cdot f. \tag{A.2.2}$$

This is because $\nabla_X v$ on the left-hand side is the covariant derivative of v along $f : M \to V$ projected down to $T(S_V)$. (For $x \in M$, $T_x(S_V)\check{}$ is the orthogonal complement of $f(x)$, and $\langle f, \check{v}\rangle = 0$.)

Proposition A.2.1. *A map $f : M \to S_V$ is harmonic iff the vector-valued function $\triangle^M f$ is a scalar multiple of f. In this case, the scalar is $e(f)$ so that we have*

$$\triangle^M f = e(f) \cdot f. \tag{A.2.3}$$

PROOF. We work out the second fundamental form $\beta(f)$. Let X and Y be vector fields on M. Then, using (A.2.1) and (A.2.2), we compute

$$\begin{aligned}
\beta(f)(X,Y)\check{} &= (\nabla_X f_*)(Y)\check{} \\
&= \nabla_X(f_*(Y))\check{} - f_*(\nabla_X Y)\check{} \\
&= X(Y(f)) - \langle X(Y(f)), f\rangle f - (\nabla_X Y)(f) \\
&= X(Y(f)) - (\nabla_X Y)(f) + \langle X(f), Y(f)\rangle f.
\end{aligned}$$

The last equality is because

$$\langle X(Y(f)), f\rangle = -\langle X(f), Y(f)\rangle \tag{A.2.4}$$

since $X\langle Y(f), f\rangle = \frac{1}{2}XY(|f|^2) = 0$. Taking traces, we obtain

$$\operatorname{trace}\beta(f)\check{} = -\triangle^M f + e(f) \cdot f.$$

Thus, if f is harmonic, then (A.2.3) holds. Conversely, if $\triangle^M f$ is a scalar multiple of f then, by sphericality of f, the scalar is $\langle \triangle^M f, f\rangle$. On the other hand, taking traces in (A.2.4) we obtain

$$\langle \triangle^M f, f\rangle = e(f).$$

The proposition follows.

Let λ be a nonzero eigenvalue of $\triangle^M$ acting on functions on M. A map $f : M \to S_V$ is said to be a λ-*eigenmap* if $\triangle^M f = \lambda f$, i.e. if the components $\alpha \circ f$, $\alpha \in V^*$, of f are eigenfunctions of $\triangle^M$ corresponding to the eigenvalue λ. (If $\lambda = \lambda_p$ is the p-th eigenvalue of $\triangle^M$ (e.g. $\lambda_p = p(p+m-1)$ for $M = S^m$) then we also say that f is a p-eigenmap.) Proposition A.2.1 says that a λ-eigenmap is nothing but a harmonic map of constant energy density λ.

We now turn to minimal immersions. Let M be a manifold of dimension m and N a Riemannian manifold. Given an immersion $f : M \to N$, we define $\wedge^m f_*$ to be the m-form on M with values in $\wedge^m f^* T(N)$ which, at $x \in M$, associates to the m-tuple $(X_x^1, \ldots, X_x^m)$ of tangent vectors at $x \in M$ the m-vector

$$(\wedge^m f_*)(X_x^1, \ldots, X_x^m) = f_*(X_x^1) \wedge \ldots \wedge f_*(X_x^m) \in \wedge^m T_{f(x)}(N).$$

The *volume density* $v(f)$ of f is the m-form on M defined as the norm of $\wedge^m f_*$ with respect to the fibre metric of $\wedge^m f^* T(N)$ (induced from the Riemannian metric on N):

$$v(f) = |\wedge^m f_*|.$$

For a precompact domain D in M, we define

$$\mathrm{vol}_D(f) = \int_D v(f)$$

as the *volume of* f *over* D. Since the volume of f is invariant under precomposition by a diffeomorphism of M (that is the identity outside a compact set), for the first variation of vol, we use (compactly supported) normal variations v of f, i.e. sections of the normal bundle ν_f of f in the orthogonal splitting

$$f^* T(N) = T(M) \oplus \nu_f.$$

The first variation formula for the volume is:

$$\delta \,\mathrm{vol}\,(v) = -\int_M \langle \,\mathrm{trace}\,\beta(f), v\rangle \cdot v_M.$$

Here $\beta(f)$ is the second fundamental form of f when M is endowed with the Riemannian metric induced from the Riemannian metric of N by the immersion f. With respect to this metric, $f : M \to N$ is an isometric immersion. The second fundamental form $\beta(f)$ takes its values in the subbundle $\nu_f \subset f^* T(N)$. This allows us to define the *second fundamental form* $\beta(f)$ of the immersion $f : M \to N$ as a symmetric 2-tensor on M with values in ν_f.

An immersion $f : M \to N$ is said to be *minimal* if its volume is stable up to first order with respect to (compactly supported) normal variations of f. The previous argument shows that an immersion is minimal iff it is harmonic as an isometric immersion. For an isometric immersion f :

$M \to N$, the energy density is clearly m so that Proposition A.2.1 gives the following:

Proposition A.2.2. *Let V be a Euclidean vector space. An immersion $f : M \to S_V$ is minimal iff*

$$\triangle^M f = m \cdot f, \tag{A.2.5}$$

where $\triangle^M$, the Laplacian on M, is taken with respect to the metric induced from S_V by f.

A converse of this is due to Takahashi [1]:

Proposition A.2.3. *Let V be a Euclidean vector space and $f : M \to V$ an isometric immersion satisfying (A.2.5). Then the image of f is contained in S_V and the restriction $f : M \to S_V$ is a minimal immersion.*

PROOF. By Proposition A.2.2, we need only prove the first statement. Let $\beta(f)$ be the second fundamental form of f. Since f maps into V, for X and Y vector fields on M, we have

$$\begin{aligned}\beta(f)(X,Y)^{\check{}} &= (\nabla_X f_*)(Y)^{\check{}} \\ &= \nabla_X(f_*(Y))^{\check{}} - f_*(\nabla_X Y)^{\check{}} \\ &= X(Y(f)) - (\nabla_X Y)(f).\end{aligned}$$

Taking traces, we obtain

$$\operatorname{trace}\beta(f)^{\check{}} = -\triangle^M f = -mf.$$

On the other hand, $\operatorname{trace}\beta(f)$ is a section of the normal bundle ν_f so that

$$\begin{aligned}\langle \operatorname{trace}\beta(f), f_*\rangle &= \langle \operatorname{trace}\beta(f)^{\check{}}, df\rangle \\ &= -\langle \triangle^M f, df\rangle = -m\langle f, df\rangle\end{aligned}$$

must vanish. We obtain that $d|f|^2 = 2\langle f, df\rangle = 0$ so that $|f|^2$ is constant. Hence the image of f is contained in a sphere of V. Again using the fact that $f : M \to V$ is isometric and (A.2.5), we have

$$\begin{aligned}0 = \triangle^M |f|^2 &= 2\langle \triangle^M f, f\rangle - 2\operatorname{trace}\langle df, df\rangle \\ &= 2m|f|^2 - 2e(f) = 2m(|f|^2 - 1),\end{aligned}$$

since $e(f) = m$. We obtain $|f|^2 = 1$, and the proposition follows.

Let $f : M \to S_V$ be an isometric minimal immersion. We define inductively the *higher fundamental forms* $\beta_k(f)$, and *osculating bundles* $\mathcal{O}^k_f$, $k = 1, \ldots, p_f$, of f on a (maximal) nonempty open set D_f as follows. For $x \in D_f$, $\beta_k(f) : S^k(T_x(M)) \to \mathcal{O}^k_{f;x}$ is a linear map of the k-th symmetric power of the tangent space $T_x(M)$ onto the fibre $\mathcal{O}^k_{f;x}$ of $\mathcal{O}^k_f$ at x. The latter is called the k-th osculating space of f at x. For $k = 1$, $\beta_1(f) = f_*$ is defined on $D^1_f = M$ and, for $x \in D^1_f$, the first osculating space $\mathcal{O}^1_{f;x}$ at x is the image of $\beta_1(f)_x$. The general induction step is given by

$$\beta_k(f)_x(X_1, \ldots, X_k) = (\nabla_{X^k}\beta_{k-1}(f))(X_1, \ldots, X_{k-1})^{\perp_{k-1}},$$

$$X_1, \ldots, X_k \in T_x(M),\ x \in D_f^{k-1},$$

where $\perp_{k-1}$ is the orthogonal projection with kernel $\mathcal{O}_{f;x}^0 \oplus \ldots \oplus \mathcal{O}_{f;x}^{k-1}$, $\mathcal{O}_{f;x}^0 = \mathbf{R} \cdot f(x)$, and D_f^k is the set of points $x \in D_f^{k-1}$ at which the image $\mathcal{O}_{f;x}^k$ of $\beta_k(f)_x$ has maximal dimension. $\beta_{p_f}(f)$ is the highest nonvanishing fundamental form and p_f is said to be the *geometric degree* of p_f. Finally,

$$D_f = \bigcap_{k=0}^{p_f} D_f^k.$$

Note that if M is analytic then so is f (by minimality) and D_f is dense in M.

A.3. Some Facts from the Representation Theory of the Special Orthogonal Group

The underlying representation theory for eigenmaps and minimal immersions of spheres is the representation theory of the special orthogonal group. Here, without completeness, we summarize some basic facts and assemble some specific formulas which are needed for the main text. For more details, see Börner [1], Fulton-Harris [1], and Knapp [1].

First recall that there is a specific choice of the maximal torus $T \subset SO(m+1)$. Namely, if

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

is counterclockwise rotation by angle $\theta \in \mathbf{R}$ on the plane, then we define the tori

$$T = \{R_{\theta_1} \oplus \ldots \oplus R_{\theta_d} \oplus [1] \,|\, \theta_j \in \mathbf{R},\ j = 1, \ldots, d\} \text{ if } m+1 = 2d+1$$

and

$$T = \{R_{\theta_1} \oplus \ldots \oplus R_{\theta_d} \,|\, \theta_j \in \mathbf{R},\ j = 1, \ldots, d\} \text{ if } m+1 = 2d.$$

Then, by Problem 1.2 (b), any $S \in SO(m+1)$ is conjugate to an element in T, so that T is a maximal torus in $SO(m+1)$. Note that, $\dim T = d = [|(m+1)/2|]$. The angular parameters allow us to view T as $\mathbf{R}^d/(2\pi\mathbf{Z})^d$. In particular, we will denote the typical element in T as $(\theta_1, \ldots, \theta_d)$.

Let V be a finite dimensional complex $SO(m+1)$-module, a complex representation space for $SO(m+1)$. Being commutative, T acts diagonally on V. Then V decomposes into the sum of *weight spaces*:

$$V = \sum_{\phi \in \Phi} V_\phi.$$

On each weight space V_ϕ, T acts by the *weight* $\phi = (\phi_1, \ldots, \phi_d) \in \mathbf{Z}^d$ as

$$v \mapsto (\theta_1, \ldots, \theta_d) \cdot v = \exp\left(i \sum_{j=1}^{d} \phi_j \theta_j\right) v, \ (\theta_1, \ldots, \theta_d) \in T, \ \ v \in V_\phi.$$

The set Φ of (nonzero) weights is finite.

The *Weyl group* $W_{SO(m+1)}$ of $SO(m+1)$ is the (finite) group of all inner automorphisms of $SO(m+1)$ that leave T invariant. We have

$$W_{SO(m+1)} \cong \mathcal{S}_d \triangleright \mathbf{Z}_2^d, \text{ if } m+1 = 2d+1$$

and

$$W_{SO(m+1)} \cong \mathcal{S}_d \triangleright \mathbf{Z}_2^{d-1}, \text{ if } m+1 = 2d,$$

where $\mathcal{S}_d$ is the symmetric group on d letters, $\mathbf{Z}_2 = \{\pm 1\}$, and $\triangleright$ is the semidirect product. For $m+1 = 2d+1$, $\sigma \in \mathcal{S}_d$ acts on T as a permutation

$$(\theta_1, \ldots, \theta_d) \mapsto (\theta_{\sigma(1)}, \ldots, \theta_{\sigma(d)})$$

and $(\epsilon_1, \ldots, \epsilon_d) \in \mathbf{Z}_2^d$ acts by

$$(\theta_1, \ldots, \theta_d) \mapsto (\epsilon_1 \theta_1, \ldots, \epsilon_d \theta_d).$$

For $m+1 = 2d$, the situation is similar, except that

$$\mathbf{Z}_2^{d-1} = \{(\epsilon_1, \ldots, \epsilon_d) \in \mathbf{Z}_2^d \mid \prod_{j=1}^{d} \epsilon_j = 1\}.$$

$W_{SO(m+1)}$ also acts on the set of weights Φ in a similar manner. Hence, if $\phi \in \Phi$ then there exists a weight $v = (v_1, \ldots, v_d) \in \mathbf{Z}^d$ in the $W_{SO(m+1)}$-orbit of ϕ satisfying

$$v_1 \geq \ldots \geq v_{d-1} \geq v_d \geq 0 \text{ if } m+1 = 2d+1 \tag{A.3.1}$$

and

$$v_1 \geq \ldots \geq v_{d-1} \geq |v_d| \text{ if } m+1 = 2d. \tag{A.3.2}$$

According to Cartan's theory, if V is irreducible then v is unique. In addition, with respect to the lexicographic order in $\mathbf{Z}^d$, v is maximal and has one-dimensional weight space V_v. We call v the *highest weight* of the complex irreducible $SO(m+1)$-module V. The highest weight v characterizes V up to equivalence, i.e., given two complex irreducible $SO(m+1)$-modules V_1 and V_2 with highest weights v_1 and v_2, then $v_1 = v_2$ implies that V_1 is equivalent to V_2 (in the sense that there exists an $SO(m+1)$-equivariant linear isomorphism between them). Moreover, for each d-tuple $v \in \mathbf{Z}^d$ satisfying (A.3.1) or (A.3.2), there is an irreducible complex $SO(m+1)$-module V such that v is the highest weight of V. It is therefore convenient to denote by V_{m+1}^v the irreducible complex $SO(m+1)$-module with highest weight

v. (We suppress the lower index when no danger of confusion arises.) As an example, computation shows that

$$V^{(p,0,\ldots,0)} = \mathcal{H}^p,$$

where $\mathcal{H}^p$ is the complex irreducible $SO(m+1)$-module of spherical harmonics of order p on S^m.

The dimension of V^v is given by the Weyl dimension formula:

$$\dim V^v_{m+1} = \frac{\prod_{i<k}(v_i - v_k - i + k)(v_k + v_k + 2d + 1 - i - k)}{\prod_{i<k}(-i+k)(2d+1-i-k)} \times \frac{\prod_{t=1}^{d}(v_t + d + 1/2 - t)}{\prod_{t=1}^{d}(d + 1/2 - t)}, \quad \text{if } m+1 = 2d+1,$$

and

$$\dim V^v_{m+1} = \frac{\prod_{i<k}(v_i - v_k - i + k)(v_i + v_k + 2d - i - k)}{\prod_{i<k}(-i+k)(2d-i-k)}, \quad \text{if } m+1 = 2d.$$

The branching rule gives the decomposition of the restriction $V^v_{m+1}|_{SO(m)}$ into irreducible components:

$$V^v_{m+1}|_{SO(m)} = \sum_{\sigma} V^{\sigma}_m, \tag{A.3.3}$$

as $SO(m)$-modules, where the summation runs over all $\sigma \in \mathbf{Z}^{|m/2|}$ for which

$$v_1 \geq \sigma_1 \geq \ldots \geq v_d \geq |\sigma_d| \text{ if } m+1 = 2d+1$$

and

$$v_1 \geq \sigma_1 \geq \ldots \geq \sigma_{d-1} \geq |v_d| \text{ if } m+1 = 2d.$$

The main purpose of this section is to describe how to decompose the tensor product

$$\mathcal{H}^p \otimes \mathcal{H}^q, \quad p \geq q,$$

into irreducible components.

First of all, for $m = 2$, the decomposition is known as the Clebsch-Gordan formula:

$$\mathcal{H}^p_2 \otimes \mathcal{H}^q_2 = \mathcal{H}^{p+q}_2 \oplus \mathcal{H}^{p+q-1}_2 \oplus (\mathcal{H}^{p-1}_2 \otimes \mathcal{H}^{q-1}_2).$$

For $m = 3$ the decomposition is still classical:

$$\mathcal{H}^p_3 \otimes \mathcal{H}^q_3 = \mathcal{H}^{p+q}_3 \oplus \sum_{r=1}^{q} (V_4^{(p+q-r,r)} \oplus V_4^{(p+q-r,-r)}) \oplus (\mathcal{H}^{p-1}_3 \otimes \mathcal{H}^{q-1}_3). \tag{A.3.4}$$

The main result of this section, then, is the following:

Theorem A.3.1. *Let $p \geq q \geq 1$. For $m \geq 4$, we have*

$$\mathcal{H}^p \otimes \mathcal{H}^q = \sum_{r=0}^{q} V^{(p+q-r,r,0,\ldots,0)} \oplus (\mathcal{H}^{p-1} \otimes \mathcal{H}^{q-1}). \tag{A.3.5}$$

Iterating (A.3.5), we obtain that

$$\mathcal{H}^p \otimes \mathcal{H}^q = \sum_{(u,v)\in\triangle_0^{p,q}\, u+v\equiv p+q \,(\mathrm{mod}\, 2)} V^{(u,v,0,\ldots,0)}, \tag{A.3.6}$$

where $\triangle_0^{p,q} \subset \mathbf{R}^2$ is the closed convex triangle with vertices $(p-q,0)$, (p,q) and $(p+q,0)$.

Remark. For m odd, $V^{(u_1,\ldots,u_d)}$ is self-conjugate iff $u_d = 0$, and for $u_d > 0$, it decomposes into the sum

$$V^{(u_1,\ldots,u_{d-1},u_d)} \oplus V^{(u_1,\ldots,u_{d-1},-u_d)}$$

of conjugate representations. In the decomposition of $\mathcal{H}^p \otimes \mathcal{H}^q$, non-self-conjugacy occurs for $m = 3$. We agree that, for $m = 3$, the symbol $V^{(u,v)}$, $v > 0$, means $V_4^{(u,v)} \oplus V_4^{(u,-v)}$. A conjugacy between $V_4^{(u,v)}$ and $V_4^{(u,-v)}$ is established by the isometry $\gamma = \mathrm{diag}\,(1,1,1,-1) \in SO(4)$. With this, (A.3.5) is also valid for $m = 3$.

In the rest of this section we give a sketch proof of Theorem A.3.1. Let $\mathcal{P} = \mathbf{C}[x_0,\ldots,x_m]$ be the ring of complex polynomials in the variables $x_0,\ldots,x_m$, and $\mathcal{P}^p \subset \mathcal{P}$ the complex linear subspace of homogeneous polynomials of degree p. $\mathcal{P}$ is a module over the general linear group $GL(m+1,\mathbf{C})$. In fact, for $g \in GL(m+1,\mathbf{C})$ and $\xi \in \mathcal{P}$, we define

$$g \cdot \xi = \xi \circ g^{-1}.$$

$\mathcal{P}^p$ is a $GL(m+1,\mathbf{C})$-submodule. Since we study representations of $SO(m+1)$, we view $\mathcal{P}$ as an $SO(m+1)$-module by restriction. Consider now the Weyl's space $\otimes^p \mathbf{C}^{m+1}$ of tensors of rank p over $\mathbf{C}^{m+1}$ of which $\mathcal{P}^p$ is an $SO(m+1)$-submodule. We now briefly recall Young's theory giving the decomposition of the traceless part $\mathcal{P}_0^p \subset \mathcal{P}^p$ into irreducible components. ($\mathcal{P}_0^p$ is comprised by those tensors whose contraction by any two indices is zero.) The symmetric group $\mathcal{S}_p$ on p letters acts on $\otimes^p \mathbf{C}^{m+1}$ by permuting the factors. This action commutes with the action of $SO(m+1)$ so that $\mathcal{S}_p$ also acts on $\mathcal{P}_0^p$. By extension, the group algebra $\mathbf{Z}_{\mathcal{S}_p}$ also acts on $\mathcal{P}_0^p$. Let $r_1 \geq \ldots \geq r_n \geq 0$ be integers with $r_1 + \ldots + r_n = p$ and consider the *Young diagram* Σ_r, $r = (r_1,\ldots,r_n)$ consisting of n rows with row lengths $r_1,\ldots,r_n$. Let $R(\Sigma_r)$ (resp. $C(\Sigma_r)$) denote the set of all permutations in $\mathcal{S}_p$ that preserve the rows (resp. columns) of Σ_r. The *Young symmetrizer* is

$$\varepsilon(\Sigma_r) = \sum_{a\in R(\Sigma_r);\, b\in C(\Sigma_r)} \mathrm{sgn}\,(b)ba \in \mathbf{Z}_{\mathcal{S}_p}.$$

A result of Young's asserts that, for each Young diagram Σ_r, the $SO(m+1)$-submodule $\varepsilon(\Sigma_r)\mathcal{P}_0^p$ of $\mathcal{P}_0^p$ is a multiple of an irreducible $SO(m+1)$-submodule of $\mathcal{P}_0^p$ and all irreducible $SO(m+1)$-submodules of $\mathcal{P}_0^p$ arise in this way. Moreover, $\varepsilon(\Sigma_r)\mathcal{P}_0^p \neq \{0\}$ iff the sum of the lengths of the first

two columns of Σ_r is $\leq m+1$. Finally, if $\Sigma_r \neq \Sigma_{r'}$ (satisfying the condition on the columns) then the corresponding irreducible $SO(m+1)$-submodules of $\mathcal{P}_0^p$ are inequivalent.

We now set $\mathcal{P} = \mathbf{C}[x_0, \ldots, x_m; y_0, \ldots, y_m]$ with $GL(m+1, \mathbf{C})$ (and hence $SO(m+1)$)-module structure given by

$$(g \cdot \xi)(x, y) = \xi(g^{-1}x, g^{-1}y),\ \xi \in \mathcal{P},\ g \in GL(m+1, \mathbf{C}),$$

where $x = (x_0, \ldots, x_m)$ and $y = (y_0, \ldots, y_m)$. The differential operators

$$\triangle_x = \sum_{i=0}^{m} \frac{\partial^2}{\partial x_i^2},\ \triangle_y = \sum_{i=0}^{m} \frac{\partial^2}{\partial y_i^2} \text{ and } D = \sum_{i=0}^{m} \frac{\partial^2}{\partial x_i \partial y_i}$$

are $SO(m+1)$-module homomorphisms of $\mathcal{P}$ with D commuting with both Laplacians. Let $\mathcal{P}^{p,q} \subset \mathcal{P}$ denote the $SO(m+1)$-submodule consisting of polynomials which are homogeneous of degree p in $x_0, \ldots, x_m$ and homogeneous of degree q in $y_0, \ldots, y_m$. We have

$$\triangle_x(\mathcal{P}^{p,q}) \subset \mathcal{P}^{p-2,q},\ \triangle_y(\mathcal{P}^{p,q}) \subset \mathcal{P}^{p,q-2} \text{ and } D(\mathcal{P}^{p,q}) \subset \mathcal{P}^{p-1,q-1}.$$

Consider the $SO(m+1)$-module:

$$H^{p,q} = \{\xi \in \mathcal{P}^{p,q} \mid \triangle_x \xi = 0,\ \triangle_y \xi = 0\}.$$

Clearly $H^{p,0} = \mathcal{H}^p$ and $H^{0,q} = \mathcal{H}^q$ as $SO(m+1)$-modules and

$$D(H^{p,q}) \subset H^{p-1,q-1}. \tag{A.3.7}$$

We claim that

$$H^{p,q} = \mathcal{H}^p \otimes \mathcal{H}^q \tag{A.3.8}$$

as $SO(m+1)$-modules. This can be seen as follows: Using (2.1.1) we write (with obvious notations):

$$\mathcal{P}^{p,0} = \mathcal{H}^p \oplus \mathcal{P}^{p-2,0}\rho_x^2 \text{ and } \mathcal{P}^{0,q} = \mathcal{H}^q \oplus \mathcal{P}^{0,q-2}\rho_y^2.$$

Substituting these into the tensor product

$$\mathcal{P}^{p,q} = \mathcal{P}^{p,0} \otimes \mathcal{P}^{0,q},$$

the only component on which both Laplacians are zero is $\mathcal{H}^p \otimes \mathcal{H}^q$. The claim follows.

From here on, without loss of generality, we assume that $p \geq q$. By (A.3.7), the differential operator D splits $H^{p,q}$ as

$$H^{p,q} = T^{p,q} \oplus V^{p,q}, \tag{A.3.9}$$

where

$$T^{p,q} = \ker(D \mid H^{p,q})$$

and

$$V^{p,q} \subset H^{p-1,q-1}. \tag{A.3.10}$$

To decompose $T^{p,q}$ into irreducible components we view

$$\mathcal{P}^{p,q} \subset \mathcal{P}^{p+q} \subset \mathbf{C}[x_0, \ldots, x_m; y_0, \ldots, y_m]$$

as an $SO(m+1)$-submodule of the Weyl's space $\otimes^{p+q}\mathbf{C}^{m+1}$. To each element $\xi \in \mathcal{P}^{p,q}$ there corresponds a tensor $\xi = (\xi_{i_1 \ldots i_{p+q}})$ of rank $p+q$ which is symmetric in $\{i_1, \ldots, i_p\}$ and $\{i_{p+1}, \ldots, i_{p+q}\}$. $\xi \in T^{p,q}$ means that $\triangle_x \xi = 0$, $\triangle_y \xi = 0$ and $D\xi = 0$ and so contraction of ξ with respect to any two indices is zero. In particular, $T^{p,q} \subset \mathcal{P}_0^{p+q}$ so that we can apply Young's theory to $T^{p,q}$.

Lemma A.3.2. *Let Σ_r be a Young diagram with row-lengths $r_1 \geq \ldots \geq r_n \geq 0$, where $r_1 + \ldots + r_n = p + q$. Then*

$$\varepsilon(\Sigma_r) T^{p,q} \neq 0$$

iff $n \leq 2$, and $0 \leq r_2 \leq q$.

PROOF. Assume, on the contrary, that $n > 2$. Since the first column of a *Young tableau* of Σ_r, the Young diagram Σ_r filled with $1, \ldots, p+q$, has at least three entries, applying a suitable permutation in $R(\Sigma_r)$, it will contain a pair (i, j) with $1 \leq i, j \leq p$ or $p+1 \leq i, j \leq p+q$. The elements of $T^{p,q}$ are symmetric in the indices i and j. On the other hand, the transposition $(ij) \in C(\Sigma_r)$ has sign -1 so that the Young symmetrizer $\varepsilon(\Sigma_r)$ annihilates $T^{p,q}$. Thus, $n \leq 2$. Assume next that $r = (r_1, r_2)$, with $r_2 > q$. A Young tableau of Σ_r, after applying a suitable element of $R(\Sigma_r)$, has entries $i_1, \ldots, i_{r_1} \leq r_1$ in the first row, and $r_1 + 1 \leq j_1, \ldots, j_{r_2} \leq r_2$ in the second row. Since $r_1 + r_2 = p + q$, $r_2 > q$ implies $r_1 < p$ so that, for some $s = 1, \ldots, r_2$, we must have $j_s \leq p$. Since $(i_s, j_s) \in C(\Sigma_r)$, it again follows that the Young symmetrizer annihilates $T^{p,q}$. The converse is clear.

According to Young's theory, the highest weight of the irreducible component in $\varepsilon(\Sigma_r)\mathcal{P}_0^{p+q}$ is $(r_1, r_2, 0, \ldots, 0)$. We obtain that

$$T^{p,q} = \sum_{r=0}^{q} m\big[V^{(p+q-r,r,0,\ldots,0)} : T^{p,q}\big] V^{(p+q-r,r,0,\ldots,0)} \tag{A.3.11}$$

where each multiplicity is positive.

PROOF OF THEOREM A.3.1. We use induction with respect to m, noting that $m = 3$ is covered by (A.3.4). To perform the general step $m \Rightarrow m+1$ we first claim that the multiplicity

$$m\big[V_{m+2}^{(p+q-r,r,0,\ldots,0)} : \mathcal{H}_{m+1}^{p} \otimes \mathcal{H}_{m+1}^{q}\big] = 1, \ r = 0, \ldots, q.$$

This follows from the induction hypothesis and the branching rule. In fact, pick the $SO(m+1)$-component $V_{m+1}^{(p+q-r,r,0,\ldots,0)}$ in the restriction $V_{m+2}^{(p+q-r,r,0,\ldots,0)}|_{SO(m+1)}$ (of multiplicity 1). By the induction hypothesis and branching, the $SO(m+1)$-module $V_{m+1}^{(p+q-r,r,0,\ldots,0)}$ is contained in

$\mathcal{H}^p_{m+1} \otimes \mathcal{H}^q_{m+1}$ exactly once. Thus

$$\begin{aligned}(1 \leq) m\big[V^{(p+q-r,r,0,\ldots,0)}_{m+2} : \mathcal{H}^p_{m+1} \otimes \mathcal{H}^q_{m+1}\big] \\ \leq m\big[V^{(p+q-r,r,0,\ldots,0)}_{m} : \mathcal{H}^p_{m+1} \otimes \mathcal{H}^q_{m+1}|_{SO(m+1)}\big] = 1,\end{aligned}$$

and we are done. Thus (A.3.11) is rewritten as

$$T^{p,q} = \sum_{r=0}^{q} V^{(p+q-r,r,0,\ldots,0)}.$$

Putting this together with (A.3.8)-(A.3.10), we obtain

$$\mathcal{H}^p \otimes \mathcal{H}^q = \sum_{r=0}^{q} V^{(p+q-r,r,0,\ldots,0)} \oplus V^{p,q},$$

where

$$V^{p,q} \subset \mathcal{H}^{p-1} \otimes \mathcal{H}^{q-1}. \tag{A.3.12}$$

It remains for us to prove that equality holds in (A.3.12). This can be shown by induction with respect to m. The general step $m \Rightarrow m+1$ is accomplished by restricting (A.3.12) to $SO(m+1)$ and working out the number of irreducible components of each side.
As a byproduct we obtain the following:

Corollary A.3.3. *For $p \geq q \geq 1$, the differential operator*

$$D : \mathcal{H}^p \otimes \mathcal{H}^q \to \mathcal{H}^{p-1} \otimes \mathcal{H}^{q-1}$$

is onto.

Remark. For a direct proof of Corollary A.3.3, see the discussion after the proof of Theorem 3.5.6, or Weingart [1] (pp. 45-46).

Corollary A.3.4. *We have*

$$S^2(\mathcal{H}^p) = \sum_{(a,b)\in\triangle^p_0;\, a,b \text{ even}} V^{(a,b,0,\ldots,0)},$$

where $\triangle^p_0 = \triangle^{p,p}_0 \subset \mathbf{R}^2$ is the closed convex triangle with vertices $(0,0)$, (p,p) and $(0,2p)$.

PROOF. $S^2(\mathcal{H}^p_m) \subset \mathcal{H}^p_m \otimes \mathcal{H}^p_m = H^{p,p} \subset \mathcal{P}^{p,p}$ corresponds to the $SO(m+1)$-module $SH^{p,p}$ of tensors ξ whose coefficients satisfy

$$\xi_{i_1\ldots i_p i_{p+1}\ldots i_{2p}} = \xi_{i_{p+1}\ldots i_{2p} i_1\ldots i_p}.$$

Also, by Corollary A.3.3, we have

$$D(SH^{p,p}) = SH^{p-1,p-1}$$

since D commutes with the symmetrization switching $\{i_1,\ldots,i_p\}$ and $\{i_{p+1},\ldots,i_{p+q}\}$. It remains for us to decompose the $SO(m+1)$-module

$$ST^{p,p} = \ker(D|SH^{p,p}) \subset \mathcal{P}^{2p}_0 \subset \otimes^{2p}\mathbf{C}^{m+1}.$$

We claim that given a Young tableau Σ_r, $r = (r_1, r_2)$, satisfying $r_1 \geq r_2 \geq 0$, $r_1 + r_2 = 2p$ and $r_2 \leq p$, then

$$\varepsilon(\Sigma_r) ST^{p,p} \neq 0$$

iff r_2 is even. Indeed, let $\xi \in ST^{p,p}$ with $\varepsilon(\Sigma_r)\xi = \xi'$. Using the symmetry properties, we compute

$$\begin{aligned}
\xi'_{i_1 \ldots i_{r_1} j_1 \ldots j_{r_2}} &= (-1)^{r_2} \xi'_{j_1 \ldots j_{r_2} i_{r_2+1} \ldots i_{r_1} i_1 \ldots i_{r_2}} \\
&= (-1)^{r_2} \xi'_{i_{p+1} \ldots i_{r_1} i_1 \ldots i_{r_2} j_1 \ldots j_{r_2} i_{r_2+1} \ldots i_p} \\
&= (-1)^{r_2} \xi'_{i_{p+1} \ldots i_{r_1} i_1 \ldots i_{r_2} i_{r_2+1} \ldots i_p j_1 \ldots j_{r_2}} \\
&= (-1)^{r_2} \xi'_{i_1 \ldots i_{r_1} j_1 \ldots j_{r_2}}.
\end{aligned}$$

Thus, for r_2 odd, $\xi = 0$. The converse also follows.

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Glossary of Notations

$\mathbf{R}^n$ n-dimensional Euclidean space

O center of mass, 3

$\{a, b\}$ Schläfli symbol of a regular polyhedron, 4

P^o reciprocal of a regular polyhedron P, 5

τ golden section, 7

$\mathcal{S}_n$ symmetric group on n letters, 10

$\mathcal{A}_n$ alternating group on n letters, 10

$G(p)$ orbit of the action of a group G through a point p, 15

G_p isotropy subgroup of the action of a group G at a point p, 17

Π set of poles of the action of a group G, 17

P_d regular d-sided polygon inscribed in the unit circle S^1, 19

C_d cyclic group of order d, 19

cyclic Möbius group, 26

D_d dihedral group of order $2d$, 19

dihedral Möbius group, 26

$O(n)$ orthogonal group, 21

$SO(n)$ special orthogonal group, 22

$\mathbf{C}^n$ complex n-space

$\hat{\mathbf{C}}$ extended complex plane, 22

$\mathcal{M}(\hat{\mathbf{C}})$ Möbius group, 22

$\mathcal{M}_0(\hat{\mathbf{C}})$ compact Möbius subgroup, 23

$SL(2, \mathbf{C})$ special linear group, 22

$SU(2)$ special unitary group, 23

S^m m-sphere

N north pole of the 2-sphere, 23

h stereographic projection, 23

$R_{\theta,x}$ rotation with axis $\mathbf{R} \cdot x$ and angle θ, 23

$\Re(z)$, $\Im(z)$ real and imaginary parts of a complex number z

$q = q_G$ (fundamental) rational function with invariance group G, 25, 46

$\mathbf{C}P^n$ complex projective n-space, 25

T tetrahedral Möbius group, 27

O octahedral Möbius group, 28

I icosahedral Möbius group, 31–32

U, V, W generators of the icosahedral group, 30–31

ω a fifth root of unity, 28

G^* binary group associated to G, 32

D_d^* binary dihedral Möbius group, 32

T^* binary tetrahedral Möbius group, 32

O^* binary octahedral Möbius group, 33

I^* binary icosahedral Möbius group, 33

$\mathbf{H}$ skew-field of quaternions, 32

T_t Clifford torus, 33

$L(p, q)$ lens space, 37

S^3/D_d^* prism manifold, 37

S^3/T^* tetrahedral manifold, 37

S^3/O^* octahedral manifold, 37

S^3/I^* icosahedral manifold, 37

ξ form, 39

χ_ξ character of a form ξ, 39

α, β, γ dihedral invariants, 41

Φ, Ψ, Ω tetrahedral invariants, 42–43

Hess Hessian, 44

Jac Jacobian, 44

$\mathcal{I}$, $\mathcal{J}$, $\mathcal{H}$ icosahedral invariants, 45–46

$\mathbf{C}[z,w]$ ring of complex polynomials in $z,w \in \mathbf{C}$, 48

$\mathbf{C}[z,w]^{G^*}$ ring of absolute invariants of G^*, 48

W_p degree p complex polynomials in $z,w \in \mathbf{C}$, 48

complex irreducible $SU(2)$-module, 51

R_{2d} real irreducible submodule of W_{2d}, 51

f_ξ orbit map through ξ, 51

L, R left and right quaternionic multiplications, 51

$\triangle$ Euclidean Laplacian, 52, 287

$\triangle^{S^m}$ spherical Laplacian, 52, 287

V Euclidean vector space, 107

S_V unit sphere in V, 107

$\mathcal{H}^p$ space of degree p harmonic polynomials on $\mathbf{R}^{m+1}$, 96

space of spherical harmonics of order p on S^m, 96

$\langle .,. \rangle_{\mathcal{H}^p}$ scalar product on $\mathcal{H}^p$, 103

$\{\xi_j\}_{j=1}^{2d+1}$ standard orthonormal basis in R_{2d}, 53

$Ver^{\mathbf{C}}$ complex Veronese map, 54

$Hopf$ Hopf map, 55

Ver_d generalized Veronese map, 55

$\mathbf{R}P^n$ real projective n-space, 56

$su(n)$ Lie algebra of $SU(n)$, 57

X, Y, Z canonical basis in $su(2)$, 57

f_* differential of a map f, 57

Tet tetrahedral minimal immersion, 60

Oct octahedral minimal immersion, 61

Ico icosahedral minimal immersion, 61

$\mathbf{H}_0$ space of pure quaternions, 62

k ground field, 68

$k(Z)$, $k(\zeta)$ fields of rational functions over k, 68

$\mathcal{S}$ Schwarzian, 69

$\tilde{\zeta}$ Tschirnhaus transformation of ζ, 70–71

δ discriminant of a polynomial, 72

$\mathcal{Q}$, $\mathcal{Q}_0$ standard projective quadric, 73, 81

P^* resolvent polynomial of P, 74

Ω_j, $j = 0, \ldots, 4$ octahedral invariants, 76

Ξ_j, $j = 0, \ldots, 4$ Hessians of octahedral invariants, 76

Υ_j, t_j, $j = 0, \ldots, 4$ roots of the canonical resolvent, 77

P^* canonical resolvent, 78

∂_x radial differentiation, 95

$\lambda_p = \lambda_{m,p}$ p-th eigenvalue of $\triangle^{S^m}$, 95

$\mathbf{R}[x_0, \ldots, x_m]$ ring of polynomials in $x_0, \ldots, x_m$, 96

$\mathcal{P}^p = \mathcal{P}^p_{m+1}$ space of degree p polynomials in $x_0, \ldots, x_m$, 96

$\langle ., . \rangle_{L^2}$ L^2 scalar product, 96

v_{S^m} volume form on S^m, 103

ρ^2 norm square in $\mathbf{R}^{m+1}$, 96

H harmonic projection operator, 98

a^* linear functional associated to $a \in \mathbf{R}^{m+1}$, 99

$o = (0, \ldots, 0, 1)$ base point in S^m, 101

$SO(m) \subset SO(m+1)$ isotropy at o, 101

∂_a directional derivative at $a \in \mathbf{R}^{m+1}$, 102

δ_a transpose of ∂_a, 102

$\iota_\pm$ imbeddings into tensor products, 103

$\mathrm{vol}\,(S^m)$ volume of S^m, 103

Γ Gamma function, 100

$c^\pm$ normalizing constants, 103–104

μ_p normalizing constant, 104

∂_ξ polynomial differential operator, 105

E_{ab} infinitesimal rotation, 106

$\iota_\wedge$ imbedding into a tensor product, 106

$\cong$ congruence, 107

V_f space of components of f, 107

$\leftharpoonup$ derivation, 108

$N(p) = N(m, p)$ dimension of $\mathcal{H}^p$ minus one, 108

$f_p = f_{m,p}$ standard minimal immersion, 108–109

$\rho_p = \rho_{m,p}$ $O(m+1)$-module structure of $\mathcal{H}^p$, 109

$S^2 V$ symmetric square of V, 110

$\odot$ symmetric tensor product, 110

$\mathcal{L}_f$ relative moduli of an eigenmap f, 111

$\mathcal{E}_f$ linear span of $\mathcal{L}_f$, 110

$\langle f' \rangle_f$ moduli point corresponding to f' in $\mathcal{L}_f$, 111

ι affine imbedding of relative moduli, 112

ι_0 linear imbedding of relative moduli, 113

$\mathcal{L}^p = \mathcal{L}^p_m$ standard moduli, 113

$\mathcal{E}^p = \mathcal{E}^p_m$ linear span of $\mathcal{L}^p$, 113

G_f eigenform of f, 119

δ Dirac delta functional, 119

$G(f)$ Gram matrix of f, 120

f^o antipodal of f, 122

λ_0 distortion function, 122

$f^\pm$ degree raising and lowering applied to f, 129

$\Phi^\pm = \Phi^\pm_p$ degree-raising and -lowering operators, 131

$\Psi^0 = \Psi^0_p$ homomorphism for sphericality, 132–133

D DoCarmo-Wallach differential operator, 138

$SU(2)'$ conjugate of $SU(2)$, 141

$(\mathcal{L}^p)^{SU(2)}$, 141$(\mathcal{L}^p)^{SU(2)'}$ equivariant moduli, 141

γ diagonal conjugation matrix, 141

$\mathcal{H}^{c,d}$ space of complex harmonic polynomials of bidegree (c, d), 141

W' $SU(2)'$-module corresponding to an $SU(2)$-module W, 142

Γ center of $U(2)$, 141

$\mathbf{P}$ projective plane in the moduli $\mathcal{L}^2_3$, 143

$\mathcal{D}$ disk in the moduli $\mathcal{L}^2_3$, 145

C^a_d ultraspherical (Gegenbauer) polynomials, 100, 149

$\tilde{f}$ domain dimension raising applied to f, 152

Θ_m domain-dimension-raising operator, 153

F quadratic polynomial map associated to a quadratic form f, 155

μ signature of F, 157

Σ_m signature space, 157

F_μ universal map for a signature μ, 158

$\mathcal{L}_\mu$ moduli for a signature μ, 158

$\mathcal{E}_\mu$ linear span of $\mathcal{L}_\mu$, 158

S^m_κ m-sphere of curvature κ, 171

$\mathcal{M}_f$ relative moduli of a minimal immersion f, 172

$\mathcal{F}_f$ linear span of $\mathcal{M}_f$, 172

$\mathcal{M}^p = \mathcal{M}^p_m$ standard moduli, 173

$\mathcal{F}^p = \mathcal{F}^p_m$ linear span of the standard moduli, 173

$V^v = V^v_{m+1}$ complex irreducible $SO(m+1)$-module with highest weight v, 177

$\triangle^{p,q}_0$ triangle with vertices $(p \pm q, 0)$, (p, q), 177

$\triangle^p_0 = \triangle^{p,p}_0$ triangle with vertices $(0,0)$, (p,p), $(2p,0)$, 177

$\triangle^p_1$ triangle with vertices $(2,2)$, (p,p), $(2(p-1),2)$, 177

$\triangle^p_2$ triangle with vertices $(4,4)$, (p,p), $(2(p-2),4)$, 179

O big O in asymptotics, 178

$\Psi(f) = \Psi_p(f)$ symmetric 2-tensor, 180

X^a conformal field with parameter $a \in \mathbf{R}^{m+1}$, 180

$\Psi(C) = \Psi_p(C)$ extension of $\Psi(f)$, 182

S^2_0 traceless symmetric square, 184

h^l l-th canonical coefficient, 187

$\beta_k(f)$ k-th fundamental form of f, 195

$\Psi^k_p(f)$ symmetric 2-tensor, 195

$\Psi^k_p(C)$ extension of $\Psi^k_p(f)$, 196

I multiindex, 197

$|I|$ number of elements in a multiindex I, 197

∂_{a_I} directional derivative parametrized by a multiindex I, 198

$\mathcal{M}^{p;k} = \mathcal{M}^{p;k}_m$ moduli for isotropic immersions, 199

$\mathcal{F}^{p;k} = \mathcal{F}^{p;k}_m$ linear span of $\mathcal{M}^{p,k}$, 199

$\widehat{\ }$ factor absent, 202

$(\mathcal{M}^p)^{SU(2)}$, $(\mathcal{M}^p)^{SU(2)'}$ equivariant moduli, 206

$\mathcal{I}$ quartic minimal immersion, 213

I, **II**, **III** congruence classes of quartic minimal immersions of type I, II, III, 217

$\mathbf{II}_0$, $\mathbf{II}_1$ subclasses of **II**, 218

$\mathcal{D}$ disk in the moduli $\mathcal{M}^4_3$, 218

$\mathcal{J}$ quartic minimal immersion, 224

Γ' maximal torus in $SU(2)$, 224

Ω projectivizing map from the moduli, 226

ι homomorphism of tensor products, 233

$\hat{f}$ infinitesimally rotated f, 241

$\mathcal{G}$ Lie algebra of a Lie group G, 241

$\mathcal{A}_p = \mathcal{A}_{m,p}$ operator of infinitesimal rotations, 242

α_p imbedding into a tensor product, 243

ad adjoint representation, 245

$[G, G]$ commutator subgroup of G, 242

Cas Casimir operator, 248

$\Lambda_p^{u,v}$ eigenvalue of $\mathcal{A}_p$ on $V^{(u,v,0,\ldots,0)}$, 248

$\mu^{u,v}$ eigenvalue of Cas on $V^{(u,v,0,\ldots,0)}$, 248

Ξ_p^k trace of a bilinear form, 250

$\mathcal{U}(\mathcal{G})$ universal enveloping algebra for $\mathcal{G}$, 264

$\mathcal{U}^d(\mathcal{G})$ space of elements in $\mathcal{U}(\mathcal{G})$ of degree $\leq d$, 264

$\mathcal{A}_f$ $\mathcal{A}_{3,p}$-invariant slice through $\langle f \rangle$, 268

U_L left-invariant extension of $U \in su(2)$, 270–271

U_R right-invariant extension of $U \in su(2)$, 270–271

$\mathcal{D}$ operator, 276

$\iota^{\mathcal{D}}$ homomorphism associated to $\mathcal{D}$, 276

$f^{\mathcal{D}}$ eigenmap associated to f, 276

$\Phi^{\mathcal{D}}$ homomorphism associated to $\mathcal{D}$, 277

Hull convex hull, 283

Ext set of extremal points, 283

λ_O distortion, 284

$e(f)$ energy density of f, 285

$E(f)$ energy of f, 285

$v(f)$ volume density of f, 289

$\mathrm{vol}(f)$ volume of f, 289

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Kannan/Krueger: Advanced Analysis
Kelly/Matthews: The Non-Euclidean Hyperbolic Plane
Kostrikin: Introduction to Algebra
Luecking/Rubel: Complex Analysis: A Functional Analysis Approach
MacLane/Moerdijk: Sheaves in Geometry and Logic
Marcus: Number Fields
McCarthy: Introduction to Arithmetical Functions
Meyer: Essential Mathematics for Applied Fields
Mines/Richman/Ruitenburg: A Course in Constructive Algebra
Moise: Introductory Problems Course in Analysis and Topology
Morris: Introduction to Game Theory
Poizat: A Course In Model Theory: An Introduction to Contemporary Mathematical Logic
Polster: A Geometrical Picture Book
Porter/Woods: Extensions and Absolutes of Hausdorff Spaces
Radjavi/Rosenthal: Simultaneous Triangularization
Ramsay/Richtmyer: Introduction to Hyperbolic Geometry
Reisel: Elementary Theory of Metric Spaces
Ribenboim: Classical Theory of Algebraic Numbers
Rickart: Natural Function Algebras
Rotman: Galois Theory
Rubel/Colliander: Entire and Meromorphic Functions
Sagan: Space-Filling Curves
Samelson: Notes on Lie Algebras
Schiff: Normal Families
Shapiro: Composition Operators and Classical Function Theory
Simonnet: Measures and Probability
Smith: Power Series From a Computational Point of View
Smith/Kahanpää/Kekäläinen/Traves: An Invitation to Algebraic Geometry
Smoryski: Self-Reference and Modal Logic
Stillwell: Geometry of Surfaces
Stroock: An Introduction to the Theory of Large Deviations
Sunder: An Invitation to von Neumann Algebras
Tondeur: Foliations on Riemannian Manifolds
Toth: Finite Möbius Groups, Minimal Immersions of Spheres, and Moduli
Wong: Weyl Transforms
Zhang: Matrix Theory: Basic Results and Techniques
Zong: Sphere Packings
Zong: Strange Phenomena in Convex and Discrete Geometry